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Extremal Point Sets and Gorenstein Ideals

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1. INTRODUCTION

The Hilbert function of a homogeneous ideal in $R=k[x_0, ..., x_n]$, k a field, is a much studied object. This is not surprising since the Hilbert function encodes important algebraic, combinatorial, and geometric information about the ideal. The fact that recent computer algebra developments have made the Hilbert function computable has not only sustained interest in them but sparked interest in many new questions about them.

In this paper, we will concentrate on the Hilbert functions which are the Hilbert functions of points in $\mathbb{P}^n$. From [11], we know that this is the same as studying 0-dimensional differentiable O-sequences (equivalently, the Hilbert functions of graded artinian quotients of $k[x_1, ..., x_n]$).

In our earlier paper [9], we began a discussion of *n-type vectors* and showed that they were in 1–1 correspondence with Hilbert functions of

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points in $\mathbb{P}^n$, a set which we denote by $\mathcal{S}_n$. This notion of *type vector* motivated our definition of *k-configurations of points in* $\mathbb{P}^n_k$. We will recall these notions in the next section.

Given any k-configuration, $\mathbb{X}$, of points in $\mathbb{P}^n$ we will show how the graded Betti numbers in the minimal free resolution of the defining ideal of $\mathbb{X}$, call it $I_{\mathbb{X}}$, does not depend on the coordinates of the points of $\mathbb{X}$ but only on the Hilbert function of $\mathbb{X}$.

We then show that for any given *type vector*, $\mathcal{T}$, we can find an example of a k-configuration, $\mathbb{X} \subset \mathbb{P}^n$, having Hilbert function $\mathbf{H}_{\mathbb{X}}$ associated to $\mathcal{T}$, and with defining ideal $I_{\mathbb{X}} \subseteq R$, which is the lift of a lex-segment artinian ideal of $S = k[x_1, ..., x_n]$. In this way we obtain that k-configurations of points in $\mathbb{P}^n$ all have the extremal resolution described by Bigatti, Hulett, and Pardue in [2, 17, 18].

Our approach describes the Betti numbers, originally found by Bigatti and Hulett, of these extremal resolutions in a new geometric way (in contrast to Bigatti's description in [2] which, following Eliahou and Kervaire [7], is combinatorial).

Using some fundamental concepts in the theory of liaison and starting with some particular k-configurations, we are able to use our results to construct families of artinian Gorenstein standard graded k-algebras all of whose graded Betti numbers can be described in terms of the *Type* of the k-configuration. This method gives all the Hilbert functions possible for codimension 3 Gorenstein artinian k-algebras.

If $\mathbf{H}$ is a possible Hilbert function of a codimension 3 artinian Gorenstein k-algebra then, from Diesel's work [4], we know that there is an extremal (maximal) resolution for homogeneous height 3 ideals $I \subseteq k[x_0, x_1, x_2] = A$ for which $B = A/I$ has Hilbert function $\mathbf{H}$. The liaison construction we referred to above (i.e., that starting with k-configurations) yields examples with this extremal resolution.

Unfortunately, there is no known analogue of Diesel's extremality result which is valid in codimensions larger than 3. However, since the liaison method we discussed above can be carried out in any codimension we have (rashly?) conjectured that there are extremal resolutions associated to the Gorenstein Hilbert functions that arise in this way and that the graded Betti numbers of those extremal resolution are the ones given by our construction!

We observe, however, that the liaison construction we are using only yields *unimodal* Gorenstein Hilbert functions (and not even all of those!) in codimension bigger than three, so our conjecture doesn't cover *all* Gorenstein Hilbert functions in those codimensions.

To give some evidence for our conjecture we prove it for the family of Gorenstein artinian graded k-algebras having the *Weak Lefschetz Property*. As J. Watanabe showed in [24], almost all Gorenstein artinian graded

k-algebras have this property, so this result makes the conjecture more interesting.

2. k-CONFIGURATIONS IN $\mathbb{P}^n$

Let $\mathcal{S}_n$ be the collection of all Hilbert functions of (reduced) points sets in $\mathbb{P}^n$. We have $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots$.

If $\mathbf{H} \in \mathcal{S}_n$ then $\Delta\mathbf{H}$ is defined by

$$\Delta\mathbf{H}(t) := \mathbf{H}(t) - \mathbf{H}(t-1).$$

It is called the *first difference of* $\mathbf{H}$.

DEFINITION 2.1. If $\mathbf{H} \in \mathcal{S}_n$ then

$$\sigma(\mathbf{H}) := \text{the least integer } t \text{ for which } \Delta\mathbf{H}(t) = 0,$$

and

$$\alpha(\mathbf{H}) := \text{the least integer } t \text{ for which } \mathbf{H}(t) < \binom{t+n}{n}.$$

Remark 2.2. Since $\mathcal{S}_i \subset \mathcal{S}_{i+1}$ there is the possibility of confusion in the definition of $\alpha(\mathbf{H})$.

For example, suppose that $\mathbf{H} := 1\ 2\ 3\ \rightarrow$. If we consider $\mathbf{H} \in \mathcal{S}_1$ then we have $\alpha(\mathbf{H}) = 3$ but if we consider $\mathbf{H} \in \mathcal{S}_2$ then we have $\alpha(\mathbf{H}) = 1$.

So, in discussing $\alpha(\mathbf{H})$ we will have to exercise care in specifying where we are considering H.

We now recall the definition of an n-*type vector*.

DEFINITION 2.3. (1) A 0-*type vector* will be defined to be $\mathcal{T} = 1$. It is the only 0-*type vector*. We shall define $\alpha(\mathcal{T}) = -1$ and $\sigma(\mathcal{T}) = 1$.

(2) A 1-*type vector* is a vector of the form $\mathcal{T} = (d)$ where $d \geqslant 1$ is a positive integer. For such a vector we define $\alpha(\mathcal{T}) = d = \sigma(\mathcal{T})$.

(3) A 2-*type vector*, $\mathcal{T}$, is

$$\mathcal{T} = ((d_1), (d_2), ..., (d_m)),$$

where $m \geqslant 1$, and the (d_i) are 1-*type vectors*. We also insist that $\sigma(d_i) < \alpha(d_{i+1})$.

For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = m$ and $\sigma(\mathcal{T}) = \sigma((d_m)) = d_m$.

Clearly, $\alpha(\mathcal{T}) \leqslant \sigma(\mathcal{T})$ with equality if and only if $\mathcal{T} = ((1), (2), ..., (m))$.

Remark. For simplicity in the notation we usually rewrite the 2-*type vector* $((d_1), ..., (d_m))$ as $(d_1, ..., d_m)$. In our earlier papers, [9, 12], we referred to this as the *alignment character*.

(4) A 3-*type vector*, $\mathcal{T}$, is an ordered collection of 2-*type vectors* $\mathcal{T}_1, ..., \mathcal{T}_r$,

$$\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_r)$$

for which $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, ..., r-1$.

For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = r$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_r)$.

(5) Now let $n \geqslant 3$. An *n-type vector*, $\mathcal{T}$, is an ordered collection of $(n-1)$-*type vectors*, $\mathcal{T}_1, ..., \mathcal{T}_s$, i.e.,

$$\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_s)$$

for which $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, ..., s-1$.

For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = s$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)$.

In [9], we proved that there is a 1–1 correspondence

$$\chi_n \colon \mathcal{S}_n \to \{n\text{-}type\ vectors\},$$

where if $\mathbf{H} \in \mathcal{S}_n$ then $\alpha(\mathbf{H}) = \alpha(\chi_n(\mathbf{H}))$ and $\sigma(\mathbf{H}) = \sigma(\chi_n(\mathbf{H}))$.
If

$$\rho_n \colon \{n\text{-}type\ vectors\} \to \mathcal{S}_n,$$

denotes the inverse to χ_n then we also showed in [9] that if $\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_r)$ is an *n-type vector* and $\mathbf{H} = \rho_n(\mathcal{T})$ then if $\tilde{\mathcal{T}} = (\mathcal{T}_1, ..., \mathcal{T}_s)$ and $\mathcal{T}' = (\mathcal{T}_{s+1}, ..., \mathcal{T}_r)$ then $\tilde{\mathcal{T}}$ and $\mathcal{T}'$ are also *n-type vectors* and if $\tilde{\mathbf{H}} = \rho_n(\tilde{\mathcal{T}})$ and $\mathbf{H}' = \rho_n(\mathcal{T}')$ then

$$\mathbf{H}(t) = \tilde{\mathbf{H}}(t-(r-s)) + \mathbf{H}'(t).$$

So, let $\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_r)$ be an *n-type vector* and suppose that $\mathcal{T}$ corresponds to $\mathbf{H} \in \mathcal{S}_n$. We want to associate to $\mathcal{T}$ (or $\mathbf{H}$) certain sets of points in $\mathbb{P}^n$ called k-configurations in $\mathbb{P}^n$ which have Hilbert function $\mathbf{H}$. We do this inductively.

Definition 2.4 (k-configuration in $\mathbb{P}^n$). ($\mathcal{S}_0$) The only element in $\mathcal{S}_0$ is $\mathbf{H} := 1 \to$. It is the Hilbert function of $\mathbb{P}^0$, which is a single point. That is the only k-configuration in $\mathbb{P}^0$.

($\mathcal{S}_1$) Let $\mathbf{H} \in \mathcal{S}_1$. Then $\chi_1(\mathbf{H}) = \mathcal{T} = (e)$ where $e \geqslant 1$. We associate to $\mathbf{H}$ any set of e distinct points in $\mathbb{P}^1$. Clearly any set of e distinct points in $\mathbb{P}^1$ has Hilbert function $\mathbf{H}$.

A set of e distinct points in $\mathbb{P}^1$ will be called a *k-configuration in* $\mathbb{P}^1$ *of type* $\mathcal{T} = (e)$.

($\mathcal{S}_2$) Let $\mathbf{H} \in \mathcal{S}_2$ and let $\mathcal{T} = ((e_1), ..., (e_r)) = \chi_2(\mathbf{H})$, where $\mathcal{T}_i = (e_i)$ is a 1-*type vector*. Choose r distinct $\mathbb{P}^1$'s in $\mathbb{P}^2$, i.e., lines in $\mathbb{P}^2$, and label them $\mathbb{L}_1, ..., \mathbb{L}_r$. By induction we choose, on $\mathbb{L}_i$, a *k-configuration* in $\mathbb{P}^1$, call it $\mathbb{X}_i$, of type $\mathcal{T}_i = (e_i)$—each *k-configuration* chosen so that no point of $\mathbb{L}_i$ contains any point of $\mathbb{X}_j$ for $j < i$.

The set $\mathbb{X} = \bigcup \mathbb{X}_i$ is called a *k-configuration in* $\mathbb{P}^2$ *of type* $\mathcal{T}$.

($\mathcal{S}_n$) ($n > 2$) Now suppose that we have defined a *k-configuration* of *Type* $\tilde{\mathcal{T}} \subseteq \mathbb{P}^{n-1}$, where $\tilde{\mathcal{T}}$ is an $(n-1)$-*type vector* associated to $G \in \mathcal{S}_{n-1}$.

Let $\mathbf{H} \in \mathcal{S}_n$ and suppose that $\chi_n(\mathbf{H}) = \mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_r)$ where the $\mathcal{T}_i$ are $(n-1)$-*type vectors*. Then $\rho_{n-1}(\mathcal{T}_i) = \mathbf{H}_i$ and $\mathbf{H}_i \in \mathcal{S}_{n-1}$.

Consider $\mathbb{H}_1, ..., \mathbb{H}_r$ distinct hyperplanes in $\mathbb{P}^n$ and let $\mathbb{X}_i$ be a *k-configuration* in $\mathbb{H}_i$ of type $\mathcal{T}_i$ such that $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for any $j < i$.

The set $\mathbb{X} = \bigcup \mathbb{X}_i$ is called a *k-configuration in* $\mathbb{P}^n$ *of type* $\mathcal{T}$. As is obvious, a *k-configuration* in $\mathbb{P}^n$ is made up of lots of different *k-configurations* in $\mathbb{P}^r$ for $r < n$, and we would like to be able to refer to these various "pieces" of the given *k-configuration.*

Notation. If $\mathbb{X}$ is a *k-configuration* of type $\mathcal{T}$ and $\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_r)$ and the $\mathbb{X}_i$ (as referred to above) are the *k-configurations* of *Type* $\mathcal{T}_i$ that make up $\mathbb{X}$, then we shall denote the $\mathbb{X}_i$ as the (first) *sub-k-configurations* of $\mathbb{X}$. Each $\mathbb{X}_i$ has its first *sub-k-configurations* and those will be called the *second sub-k-configurations* of $\mathbb{X}$. We continue in this fashion as needed.

Remark 2.5. It follows from [9] that the Hilbert function of a k-configuration in $\mathbb{P}^n$ is determined by the Hilbert function of its sub-k-configurations via the following "addition" formula,

$$\begin{aligned}\mathbf{H}(\mathbb{X}, j) &= \mathbf{H}(\mathbb{X}_1, j-(u-1)) + \cdots + \mathbf{H}(\mathbb{X}_u, j) \\ &= \sum_{i=1}^{u} \mathbf{H}(\mathbb{X}_i, j-(u-i))\end{aligned}$$

for any j.

Remark 2.6. Let $I_{\mathbb{X}}$ be the ideal of a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$. Furthermore let $d_1, ..., d_t$ be the degrees of the minimal generators of $I_{\mathbb{X}}$. We denote by $\Delta(I_{\mathbb{X}})$ the set $\{d_1, ..., d_t\}$. This is somewhat unorthodox since some of the d_i's might be equal to each other. Also, for an integer d, we denote by $\Delta(I_{\mathbb{X}})+d$ the set $\{d_1+d, ..., d_t+d\}$.

THEOREM 2.7. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ of type $\mathcal{T}=(\mathcal{T}_1, ..., \mathcal{T}_u)$ and $\mathbb{X}_1, ..., \mathbb{X}_u$ be the first sub-k-configurations of $\mathbb{X}$. Then*

$$v(I_{\mathbb{X}})=v(\bar{I}_{\mathbb{X}_1})+\cdots+v(\bar{I}_{\mathbb{X}_j})+\cdots+v(\bar{I}_{\mathbb{X}_u})+1 \tag{2.1}$$

and

$$\Delta(I_{\mathbb{X}})=\{u, \Delta(\bar{I}_{\mathbb{X}_1})+u-1, ..., \Delta(\bar{I}_{\mathbb{X}_j})+u-j, ..., \Delta(\bar{I}_{\mathbb{X}_u})\}, \tag{2.2}$$

where $v(I)$ is the number of the minimal generators of I and

$$\bar{I}_{\mathbb{X}_j}=[I_{\mathbb{X}_j}+I_{\mathbb{H}_j}]/I_{\mathbb{H}_j}\subset R/I_{\mathbb{H}_j}$$

for every $1\leqslant j\leqslant u$.

Proof. We shall prove this theorem by double induction on n and u. The theorem holds by Theorem 2.6 in [12] and Theorem 2.5 in [21] when $n=2$ or 3. Assume $n>3$. Let $\mathbb{H}_i$ be a hyperplane in $\mathbb{P}^n$ which contains $\mathbb{X}_i$ for every i. If $u=1$, then $\mathbb{X}$ is a k-configuration in $\mathbb{P}^{n-1}$ of type $\mathcal{T}_1$. Let $I_{\mathbb{H}_1}=(H_1)$, $S=R/(H_1)$ and $J_1=(I_{\mathbb{X}}+(H_1))/(H_1)$ $(=I_{\mathbb{X}}/(H_1))$. Then J_1 is the ideal of a k-configuration in $\mathbb{P}^{n-1}$ of type $\mathcal{T}_1$. Hence, by induction on n, there exist F's $\in I_{\mathbb{X}}$ with degrees

$$\{\Delta(\bar{I}_{\mathbb{X}_1})\}$$

such that $J_1=\langle \bar{F}\text{'s}\rangle$. Hence

$$I_{\mathbb{X}}=\langle H_1, F\text{'s}\rangle,$$

and this proves the theorem when $u=1$.

Now suppose $u>1$. Let $\mathbb{Y}=\bigcup_{i=1}^{u-1}\mathbb{X}_i$. Then $\mathbb{Y}$ is a k-configuration in $\mathbb{P}^n$ of type $\mathcal{T}=(\mathcal{T}_1, ..., \mathcal{T}_{u-1})$. Hence

$$v(I_{\mathbb{Y}})=v(\bar{I}_{\mathbb{X}_1})+\cdots+v(\bar{I}_{\mathbb{X}_j})+\cdots+v(\bar{I}_{\mathbb{X}_{u-1}})+1$$

and the degrees of the minimal generators of $I_{\mathbb{Y}}$ are

$$\Delta(I_{\mathbb{Y}})=\{u-1, \Delta(\bar{I}_{\mathbb{X}_1})+(u-1)-1, ..., \Delta(\bar{I}_{\mathbb{X}_j})+(u-1)-j, ..., \Delta(\bar{I}_{\mathbb{X}_{u-1}})\}$$

by induction on u. Let $I_{\mathbb{H}_u}=(H_u)$, $T=R/(H_u)$, and $J_2=(I_{\mathbb{X}}+(H_u))/(H_u)$. Then

$$\frac{I_{\mathbb{X}}}{H_u\cdot[I_{\mathbb{X}}:H_u]}=\frac{I_{\mathbb{X}}}{(H_u)\cap I_{\mathbb{X}}}\simeq\frac{I_{\mathbb{X}}+(H_u)}{(H_u)}=J_2\subset T.$$

Thus we have an exact sequence of graded modules

$$0 \to [I_{\mathbb{X}} : H_u](-1) \xrightarrow{\times H_u} I_{\mathbb{X}} \to \underset{\displaystyle J_2}{\underset{\|}{\frac{I_{\mathbb{X}} + (H_u)}{(H_u)}}} \to 0. \tag{2.3}$$

Let

$$\mathbb{Y} = \{P_1, ..., P_s\} \qquad \text{and} \qquad \mathbb{X}_u = \{P_{s+1}, ..., P_{s+t}\},$$

$$\wp_i = I_{\mathbb{X}}(P_i), \qquad \text{for every} \quad i = 1, ..., s+t.$$

Since

$$[\wp_i : H_u] = \begin{cases} R, & \text{if} \quad H_u \in \wp_i \\ \wp_i, & \text{if} \quad H_u \notin \wp_i, \end{cases}$$

we have, for every $i = 1, ..., s+t$,

$$[I_{\mathbb{X}} : H_u] = \left[\bigcap_{i=1}^{s+t} \wp_i : H_u\right] = \bigcap_{i=1}^{s+t} [\wp_i : H_u] = \bigcap_{i=1}^{s} [\wp_i : H_u] = \bigcap_{i=1}^{s} \wp_i = I_{\mathbb{Y}}.$$

Thus we can rewrite the exact sequence (2.3) as

$$0 \to I_{\mathbb{Y}}(-1) \xrightarrow{\times H_u} I_{\mathbb{X}} \to J_2 \to 0. \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned} \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{H}_u}), t) &= \mathbf{H}(T/J_2, t) \\ &= \begin{cases} 1, & \text{for} \quad t = 0 \\ \mathbf{H}(R/I_{\mathbb{X}}, t) - \mathbf{H}(\mathbb{Y}, t-1), & \text{for} \quad t \geqslant 1, \end{cases} \\ &= \mathbf{H}(\mathbb{X}_u, t), \end{aligned}$$

which implies J_2 is a saturated ideal, i.e., $I_{\mathbb{X}} + (H_u) = I_{\mathbb{X}_u}$.

By induction on n, there exist F's $\in I_{\mathbb{X}}$ with degrees

$$\{\Delta(\bar{I}_{\mathbb{X}_u})\}$$

such that $\bar{F}$'s are the minimal generators of J_2. Let $\{G\text{'s}\}$ be the minimal generators of $I_{\mathbb{Y}}$ and $\{K\text{'s}\} = \{G \cdot H_u\text{'s}\} \cup \{F\text{'s}\}$.

Claim. $I_{\mathbb{X}} = \langle K\text{'s} \rangle$.

Proof of Claim. Clearly, $\langle K\text{'s} \rangle \subseteq I_{\mathbb{X}}$. Conversely, for every $F \in I_{\mathbb{X}}$, $\bar{F} \in J_2$. Hence

$$F = \left[\sum F\text{'s} \cdot N\text{'s} \right] + H_u M$$

for some N's, $M \in R$. Since $M \in [I_{\mathbb{X}} : H_u] = I_{\mathbb{Y}}$,

$$M = \sum G\text{'s} \cdot Q\text{'s}$$

for some Q's $\in R$. Hence

$$\begin{aligned} F &= \left[\sum F\text{'s} \cdot N\text{'s} \right] + H_u M \\ &= \left[\sum F\text{'s} \cdot N\text{'s} \right] + H_u \left[\sum G\text{'s} \cdot Q\text{'s} \right] \\ &= \left[\sum F\text{'s} \cdot N\text{'s} \right] + \sum [G \cdot H_u]\text{'s} \cdot Q\text{'s} \\ &\in \langle K\text{'s} \rangle. \end{aligned}$$

Hence we are done. ∎

Note that if we let $\tilde{\mathbb{X}}_j$ denote the k-configuration $\mathbb{X}_j$, but considered in $\mathbb{P}^{n-1}$, then $\Delta(\bar{I}_{\mathbb{X}_j}) = \Delta(I_{\tilde{\mathbb{X}}_j})$.

Remark 2.8. Theorem 2.7 makes two things very clear. The first is that the number of minimal generators of $I_{\mathbb{X}}$, and their degrees, do not depend on $\mathbb{X}$ but only on *Type*($\mathbb{X}$). This follows once we observe: (i) the inductive nature of the description of these values and (ii) the fact that this is true for *k-configurations* in $\mathbb{P}^2$ (see Theorem 2.6 in [12]). Also, it is clear how important the *tree* of sub-*k-configurations* of $\mathbb{X}$ really is.

We use this theorem as motivation for describing the *rooted tree*, $T(\mathscr{T})$, associated to an *n-type vector* $\mathscr{T}$. The definition will be made inductively.

Definition 2.9. If $\mathscr{T}$ is a 0-*type vector* then $T(\mathscr{T}) = \varnothing$.

If $\mathscr{T} = (e)$ is a 1-*type vector* then $T(\mathscr{T})$ has one node, and no edges. That node is the root of the tree.

If $\mathscr{T} = (\mathscr{T}_1, ..., \mathscr{T}_u)$ is an *n-type vector* and we have already defined the rooted trees $T(\mathscr{T}_1)$, ..., $T(\mathscr{T}_u)$, then $T(\mathscr{T})$ is formed by taking one new node,

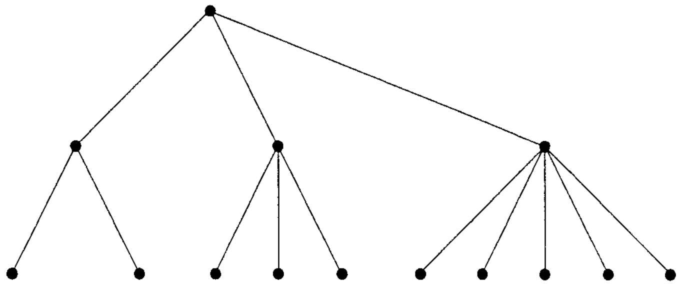

FIGURE 1

which will be the root of $T(\mathscr{T})$, and u new edges which connect this new node to the roots of the trees $T(\mathscr{T}_1), ..., T(\mathscr{T}_u)$.

EXAMPLE 2.10. Let $\mathscr{T} = ((1, 2); (1, 3, 4); (2, 3, 4, 5, 6))$ be a 3-*type vector*. The rooted tree $T(\mathscr{T})$ is shown in Fig. 1.

The usefulness of this tree is apparent from:

PROPOSITION 2.11. *Let $\mathscr{T}$ be an n-*type vector, *$n \geqslant 2$, and let $\mathbb{X}$ be a k-*configuration *in $\mathbb{P}^n$ of* type *$\mathscr{T}$ Let $T(\mathscr{T})$ be the rooted tree associated to $\mathscr{T}$ and let $I_{\mathbb{X}} \subset k[x_0, ..., x_n]$ be the ideal of $\mathbb{X}$.*

Then the minimal number of generators of $I_{\mathbf{X}}$ is the number of nodes in the tree $T(\mathscr{T})$.

Proof. This is obvious from the inductive description of $\nu(I_{\mathbb{X}})$ in Theorem 2.7, the inductive definition of $T(\mathscr{T})$ and the fact that the equality is immediate for $n = 1$. ∎

Remark 2.12. (1) If $\mathbb{X}$ is a *k-configuration* of type $\mathscr{T}$, ($\mathscr{T}$ as in Example 2.10), then we always have $\nu(I_{\mathbb{X}}) = 14$ since the tree $T(\mathscr{T})$ has 14 nodes.

(2) Notice that for a *k-configuration* in $\mathbb{P}^2$ this result gives that $\nu(I_{\mathbb{X}}) = \alpha(I_{\mathbb{X}}) + 1$, the classic bound of P. Dubreil for the minimal number of generators of codimension 2 Cohen Macaulay subschemes of $\mathbb{P}^N$ (see [5]).

Now for *k-configurations* in $\mathbb{P}^n$ ($n \geqslant 2$), one can easily prove that if we let $\mathscr{T}_1, ..., \mathscr{T}_N$ denote the type vectors associated to every node in the tree $T(\mathscr{T})$ except the "leaves" of the tree, then the number of nodes in the tree is nothing more than

$$\sum_{i=1}^{N} \alpha(\mathscr{T}_i) + 1.$$

Thus, once we prove (see Theorem 4.4) that *k-configurations* have maximal graded Betti numbers among the arithmetically Cohen–Macaulay subschemes of $\mathbb{P}^n$ with given Hilbert function, we will have expressed the upper bound for the number of generators of codimension n Cohen–Macaulay subschemes of $\mathbb{P}^r$ having that Hilbert function in a "Dubreil-like" manner.

Remark 2.13. (1) The rooted tree $T(\mathscr{T})$ associated to the *n-type vector* $\mathscr{T}$, has some special properties. To describe these special properties we introduce the following notion: a node of $T(\mathscr{T})$ has *distance t from the root of* $T(\mathscr{T})$, if one must traverse exactly t edges to get to that node from the root. The set of nodes at distance t from the root will be denoted $T(\mathscr{T})_t$. All nodes have distance less than or equal to $n-1$ from the root and all the "leaves" of the rooted tree have distance exactly $n-1$ from the root.

(2) We now observe that there is a natural total ordering on the set $T(\mathscr{T})_t$. We describe this ordering inductively.

$t=1$. Since $\mathscr{T}=(\mathscr{T}_1, ..., \mathscr{T}_r)$, the set $T(\mathscr{T})_1$ corresponds to the $(n-1)$-*type vectors* $\mathscr{T}_1, ..., \mathscr{T}_r$ and so we order the nodes in this set according to this ordering.

Now suppose that we have ordered the nodes in all the rooted trees $T(\mathscr{T})$ where $\mathscr{T}$ is an *i-type vector* for $i<n$ and all the nodes in $T(\mathscr{T})_j$ for $j<t$, t fixed, in all rooted trees corresponding to *n-type vectors* $\mathscr{T}$. We now seek to order the nodes in $T(\mathscr{T})_t$ where $\mathscr{T}$ is an *n-type vector*.

Let A and B be two nodes in $T(\mathscr{T})_t$. Then there are unique nodes N_A and N_B in $T(\mathscr{T})_{t-1}$ to which A and B are connected. If $N_A=N_B$ then we can consider the tree beginning at that node. It is a rooted tree corresponding to an $(n-(t-1))$-*type vector* and so we are done by induction.

If $N_A \neq N_B$ then we can assume, with no loss of generality, that $N_A<N_B$. We set $A<B$.

Now notice that if we choose two nodes, A and B in $T(\mathscr{T})_{i-1}$ and look at all the nodes in $T(\mathscr{T})_i$ connected to A (call it $\mathscr{J}B(A)-\mathscr{J}B$ for "just below") and to B (call it $\mathscr{J}B(B)$), then if $A<B$, we have $|\mathscr{J}B(A)|<|\mathscr{J}B(B)|$. This is a special aspect of the rooted trees that arise from *n-type vectors.*

(3) If we are willing to put labels on the edges and nodes of $T(\mathscr{T})$ then we can also use it to find the degrees in a set of minimal generators of $I_{\mathbb{X}}$. We won't use this procedure in this paper for any proofs but it is so useful for examples that we thought it cruel not to describe it, even if our description is somewhat informal.

The labels we'll place on the edges of the tree will be numbers and we'll think of "buckets" placed at each node, into which we will toss numbers.

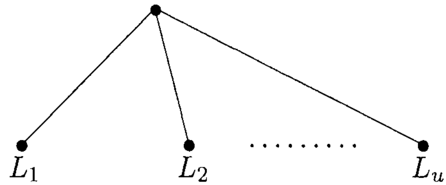

FIG. 2. Subtree of A.

We start the process by first putting, in the bucket at node A, the number $|\mathcal{J}B(A)|$. That puts a number in every bucket except those at the "leaves" of the tree. So, we now describe what to put in those buckets.

Let A be a node at distance $n-2$ from the root and consider $\mathcal{J}B(A) = \{L_1, ..., L_u\}$. Now A corresponds to a 2-*type vector*, i.e.,

$$A \leftrightarrow \mathcal{T}' = ((e_1), ..., (e_u)).$$

So we have the subtree of $T(\mathcal{T})$ shown in Fig. 2.

In the bucket at leaf L_i, we place the number e_i. Now every bucket has a number in it.

Now we label the edges of $T(\mathcal{T})$. Let $A \in T(\mathcal{T})_{i-1}$, then the nodes in $\mathcal{J}B(A)$ are ordered by $B_1, ..., B_s$. We place the number $s-i$ on the edge connecting A to B_i. In this way we label all the edges in the tree.

Finally we describe an algorithm which converts entries from the buckets on the ith level into entries for the buckets on the $(i-1)$st level. The algorithm works as follows: If $B \in \mathcal{J}B(A)$ and the edge connecting B to A has the label r on it, the we add r to every element in the bucket at B and toss the result in the bucket at A (we allow repeated numbers to appear in a bucket). The algorithm begins at the $n-1$ level.

The algorithm finishes with a great many numbers in the "root bucket;" those numbers are the degrees in a minimal generating set for $I_{\mathbb{X}}$ where $\mathbb{X}$ is a *k-configuration* in $\mathbb{P}^n$ of *Type* $\mathcal{T}$.

EXAMPLE 2.14. We illustrate the procedure continuing with Example 2.10 from above. Before the algorithm begins the tree is labeled as shown in Fig. 3.

If we let the algorithm work on the bottom level it deposits

$1+1, 2+0$	in bucket A_{11};
$1+2, 3+1, 4+0$	in bucket A_{12};
$2+4, 3+3, 4+2, 5+1, 6+0$	in bucket A_{13}.

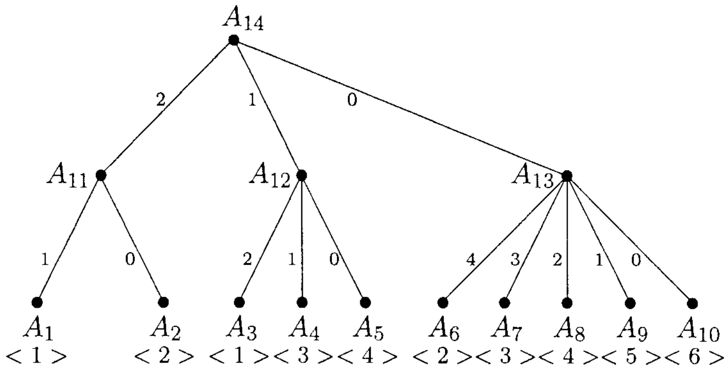

FIGURE 3

Ignoring the bottom layer we now have

$$A_{11} \text{ contributes } 2+2,\ 2+2,\ 2+2 \text{ to bucket } A_{14};$$

$$A_{12} \text{ contributes } 3+1,\ 3+1,\ 4+1,\ 4+1 \text{ to bucket } A_{14};$$

while

$$A_{13} \text{ contributes } 5+0,\ 6+0,\ 6+0,\ 6+0,\ 6+0,\ 6+0 \text{ to bucket } A_{14}.$$

So, bucket A_{14} ends up with

$$\langle \underbrace{3}, \underbrace{4, 4, 4}, \underbrace{4, 4, 5, 5}, \underbrace{5, 6, 6, 6, 6, 6} \rangle$$

and these are the degree of the elements in a minimal generating set for $I_{\mathbb{X}}$ where $\mathbb{X}$ is any *k-configuration* in $\mathbb{P}^3$ of *Type* $\mathscr{T}$ (see Fig. 4).

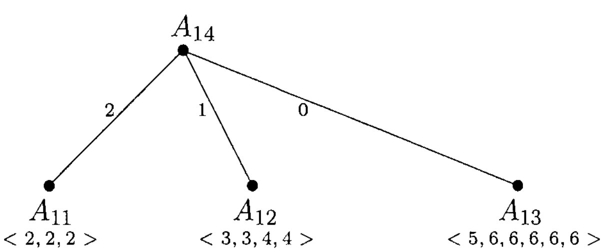

FIGURE 4

3. THE MINIMAL FREE RESOLUTION OF THE IDEAL OF A k-CONFIGURATION IN $\mathbb{P}^n$

In [12, 13], we gave a complete description of the Betti numbers in the minimal free resolution of a k-configuration in $\mathbb{P}^2$ and $\mathbb{P}^3$. That description depended only on the *Type* of the k-configuration. We wish to extend those results to $\mathbb{P}^n$ for all n.

So, let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ $(n \geqslant 4)$. Then $I_{\mathbb{X}}$ has a resolution,

$$0 \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \mathscr{F}_j \to \cdots \to \mathscr{F}_1 \to R \to R/I_{\mathbb{X}} \to 0.$$

Theorem 2.7 gives a complete description of $\mathscr{F}_1$, i.e., a complete description of the first graded Betti numbers of $\mathbb{X}$. We noted in Remark 2.8 that this description doesn't depend on the coordinates of the points of $\mathbb{X}$ but only on its *Type*.

Before entering into the proof of our theorem about the resolutions of the defining ideals of k-configurations, we will gather together some simple remarks.

LEMMA 3.1. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ of type $\mathscr{T} = (\mathscr{T}_1, ..., \mathscr{T}_u)$. The highest degree of a minimal generator of $I_{\mathbb{X}}$ is always $\sigma(\mathbb{X}) = \sigma(\mathscr{T})$.*

Proof. By very general considerations we know that the degrees of all the minimal generators of $I_{\mathbb{X}}$ are $\leqslant \sigma(\mathbb{X})$. Also, the equality of the lemma is obvious for $n=1$ and proved in Theorem 2.6 of [12] for $n=2$. We proceed by induction.

We can thus assume that any k-configuration in $\mathbb{P}^{n-1}$ of type $\mathscr{T}_u$ has a minimal generator of degree $\sigma(\mathscr{T}_u)$. In particular this is true for the first sub k-configuration of $\mathbb{X}$ which is $\mathbb{X}_u$; i.e., $\sigma(\mathscr{T}_u) = \sigma(\mathbb{X}_u) \in \Delta(\bar{I}_{\mathbb{X}_u})$. By definition $\sigma(\mathscr{T}) = \sigma(\mathscr{T}_u)$ and we proved that $\sigma(\mathscr{T}) = \sigma(\mathbb{X})$ in [9] and so by Theorem 2.7 we get that there is a minimal generator of $\mathbb{X}$ having degree $\sigma(\mathbb{X})$. ∎

With that observation out of the way we can say the following:

PROPOSITION 3.2. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ of type $\mathscr{T} = (\mathscr{T}_1, ..., \mathscr{T}_u)$ and let $\mathbb{X}_1, ..., \mathbb{X}_u$ be the first sub-k-configurations of $\mathbb{X}$.*

Let

$$a_{j0} \leqslant a_{j1} \leqslant \cdots \leqslant a_{jl_j}$$

be the elements of the family $\Delta(\bar{I}_{\mathbb{X}_j})+u-j$. *Then*

$$a_{jl_j} \leqslant a_{(j+1)0}$$

for every $a \leqslant j \leqslant u-1$.

Proof. The result will follow from the following.

CLAIM. $a_{j0} = \alpha(\mathbb{X}_j)+(u-j)$ *and* $a_{jl_j} = \sigma(\mathbb{X}_j)+(u-j)$.

Proof of Claim. Clearly $\alpha(\mathbb{X}_j)$ is the smallest integer in $\Delta(\bar{I}_{\mathbb{X}_j})$, so the first equality is obvious. By Lemma 3.1, the second is also clear.

Now, $\sigma(\mathbb{X}_j) < \alpha(\mathbb{X}_{j+1})$, so

$$\sigma(\mathbb{X}_j)+(u-j) \leqslant \alpha(\mathbb{X}_{j+1})+(u-j)-1,$$

i.e., using the claim, this means $a_{jl_j} \leqslant a_{(j+1)0}$, as was to be shown. ∎

Remark 3.3. It is possible, in general, to describe a_{j1}; one can see (again, by induction) that

$$a_{j1} = \alpha(\mathbb{X}_{(j,1)})+\alpha(\mathbb{X}_j)+(u-j-1),$$

where $\mathbb{X}_{(j,1)}$ is the first of the first sub-k-configurations of $\mathbb{X}_j$.

We now want to give a description of all the graded Betti numbers in a minimal free resolution of a k-configuration of points in $\mathbb{P}^n$. This will be done inductively and will depend only on the *Type* of the configuration.

We proceed by induction on the numbers of hyperplanes in the configuration ("u" in the definition). We start with the case $u=1$.

Let $R=k[x_0, ..., x_n]$ and $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ ($n \geqslant 4$) of type $\mathcal{T}=(\mathcal{T}_1)$ where $\mathcal{T}_1$ is an $(n-1)$-type vector. So $\mathbb{X}=\mathbb{X}_1$ is a k-configuration in a hyperplane $\mathbb{H}$ of $\mathbb{P}^n$. Without loss of generality we may assume $I_{\mathbb{H}}=(x_n)$. Let

$$J=(I_{\mathbb{X}}+(x_n))/(x_n)=I_{\mathbb{X}}/(x_n) \subset S=k[x_0, ..., x_n]/(x_n).$$

LEMMA 3.4. *Let* R, S, J, *and* $\mathbb{X}$ *be as above and let*

$$0 \to \mathcal{F}'_{n-1} \to \cdots \to \mathcal{F}'_1 \to S \to S/J \to 0$$

be the minimal free resolution of S/J *as an* S*-module. Set* $T=k[x_n]$. *Then the minimal free resolution of* $R/I_{\mathbb{X}}$, *as an* R*-module, is*

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_j \to \cdots \to \mathcal{F}_1 \to R \to R/I_{\mathbb{X}} \to 0,$$

where

$$\mathscr{F}_1 = [\mathscr{F}'_1 \otimes_k T] \oplus [S \otimes_k T(-1)],$$
$$\vdots$$
$$\mathscr{F}_j = [\mathscr{F}'_j \otimes_k T] \oplus [\mathscr{F}'_{j-1} \otimes_k T(-1)],$$
$$\vdots$$
$$\mathscr{F}_{n-1} = [\mathscr{F}'_{n-1} \otimes_k T] \oplus [\mathscr{F}'_{n-2} \otimes_k T(-1)],$$

and

$$\mathscr{F}_n = [\mathscr{F}'_{n-1} \otimes_k T(-1)].$$

Proof. Assume the minimal free resolution of S/J is

$$0 \to \mathscr{F}'_{n-1} \to \cdots \to \mathscr{F}'_1 \to S \to S/J \to 0.$$

Since the T-resolution of $k = T/(x_n)$ is

$$\begin{array}{c} 0 \to T(-1) \xrightarrow{\times x_n} T \to k \to 0, \\ \| \\ T/(x_n) \end{array}$$

the ideal $I_{\mathbb{X}}$ of $\mathbb{X}$ considered in $\mathbb{P}^n$ is given by

$$(J \otimes_k T) + (S \otimes_k (x_n)) = I_{\mathbb{X}}$$

in $S \otimes_k T = R$ and it is well-known that a minimal free resolution of such an $I_{\mathbb{X}}$ is given by tensoring the minimal resolutions of S/J and k. Hence the minimal free resolution of $R/I_{\mathbb{X}}$ is

$$\begin{aligned} 0 &\to \mathscr{F}'_{n-1} \otimes_k T(-1) \to [\mathscr{F}'_{n-1} \otimes_k T] \oplus [\mathscr{F}'_{n-2} \otimes_k T(-1)] \\ &\to \cdots \to [\mathscr{F}'_i \otimes_k T] \oplus [\mathscr{F}'_{i-1} \otimes_k T(-1)] \\ &\to \cdots \to [\mathscr{F}'_1 \otimes_k T] \oplus [S \otimes_k T(-1)] \to S \otimes_k T \\ &\to [S \otimes_k T]/[J \otimes_k T + S \otimes_k (x_n)] \to 0. \end{aligned} \tag{3.1}$$

Since $R = S \otimes_k T$ and $(J \otimes_k T) + (S \otimes_k (x_n)) = I_{\mathbb{X}}$, we can rewrite (3.1) as

$$\begin{aligned} 0 &\to \mathscr{F}'_{n-1} \otimes_k T(-1) \to [\mathscr{F}'_{n-1} \otimes_k T] \oplus [\mathscr{F}'_{n-2} \otimes_k T(-1)] \\ &\to \cdots \to [\mathscr{F}'_j \otimes_k T)] \oplus [\mathscr{F}'_{i-1} \otimes_k T(-1)] \to \cdots \\ &\to [\mathscr{F}'_1 \otimes_k T] \oplus [S \otimes_k T(-1)] \to R \to R/I_{\mathbb{X}} \to 0. \end{aligned} \tag{3.2}$$

This completes the proof of the lemma. ∎

EXAMPLE 3.5. Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^2$ of *Type* (2, 3, 4, 5, 6). Then a minimal free resolution of $I_{\mathbb{X}}$ as $S = k[x_0, x_1, x_2]$-module is

$$0 \to S(-7)^5 \to S(-6)^5 \oplus S(-5) \to I_{\mathbb{X}} \to 0.$$

On the other hand if we consider a k-configuration in $\mathbb{P}^3$ of *Type* ((2, 3, 4, 5, 6)) then a minimal free resolution of $I_{\mathbb{X}}$ as an $R = k[x_0, x_1, x_2, x_3]$-module is

$$0 \to R(-8)^5 \to R(-7)^{10} \oplus R(-6) \to R(-6)^5 \oplus R(-5) \oplus R(-1) \to I_{\mathbb{X}} \to 0.$$

We are now ready to state the main theorem of this paper.

Let $\mathbb{X}$, $\mathbb{X}_i$, and $\mathbb{H}_i$ be as in Definition 2.4. Let $\mathbb{Y} := \bigcup_{i=1}^{u-1} \mathbb{X}_i$. Then $\mathbb{Y}$ is also a k-configuration in $\mathbb{P}^n$. We shall prove the following theorem using induction on u. First, assume

$$0 \to \mathscr{D}_n \to \mathscr{D}_{n-1} \to \cdots \to \mathscr{D}_j \to \cdots \to \mathscr{D}_1 \to R \to R/I_{\mathbb{Y}} \to 0$$

and

$$0 \to \mathscr{E}_n \to \mathscr{E}_{n-1} \to \cdots \to \mathscr{E}_j \to \cdots \to \mathscr{E}_1 \to R \to R/I_{\mathbb{X}_u} \to 0$$

are minimal free resolutions of $R/I_{\mathbb{Y}}$ and $R/I_{\mathbb{X}_u}$, respectively.

THEOREM 3.6. *Let $\mathbb{X}$, $\mathbb{X}_i$, $\mathbb{H}_i$, $\mathbb{Y}$, $\mathscr{D}_j$, and $\mathscr{E}_j$ be as above and let a_{ji} be as in Lemma* 3.2. *Then the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \to \mathscr{F}_j \to \cdots \to \mathscr{F}_1 \to R \to R/I_{\mathbb{X}} \to 0,$$

where

$$\begin{aligned}
\mathscr{F}_1 &= \mathscr{D}_1(-1) \oplus \mathscr{E}_1/R(-1) \\
\mathscr{F}_2 &= \mathscr{D}_2(-1) \oplus \mathscr{E}_2 \\
&\ \vdots \\
\mathscr{F}_j &= \mathscr{D}_j(-1) \oplus \mathscr{E}_j \\
&\ \vdots \\
\mathscr{F}_n &= \mathscr{D}_n(-1) \oplus \mathscr{E}_n.
\end{aligned}$$

Proof. If $u = 1$, then we are done by Lemma 3.4. Assume $u > 1$. By Theorem 2.7 applied to $\mathbb{X}$ and $\mathbb{Y}$ and Lemma 3.4 applied to $\mathbb{X}_u$, we get that

$$\mathscr{F}_1 = \mathscr{D}_1(-1) \oplus \mathscr{E}_1/R(-1).$$

Let $S = R/I_{\mathbb{H}_u}$ and $J = [I_{\mathbb{X}} + I_{\mathbb{H}_u}]/I_{\mathbb{H}_u} \subset S$. Then

$$\frac{I_{\mathbb{X}}}{I_{\mathbb{H}_u} \cdot [I_{\mathbb{X}} : I_{\mathbb{H}_u}]} = \frac{I_{\mathbb{X}}}{I_{\mathbb{H}_u} \cap I_{\mathbb{X}}} \simeq \frac{I_{\mathbb{X}} + I_{\mathbb{H}_u}}{I_{\mathbb{H}_u}} = J \subset S.$$

Thus we have an exact sequence of graded modules:

$$0 \to [I_{\mathbb{X}} : I_{\mathbb{H}_u}](-1) \xrightarrow{\times I_{\mathbb{H}_u}} I_{\mathbb{X}} \to \frac{I_{\mathbb{X}} + I_{\mathbb{H}_u}}{I_{\mathbb{H}_u}} \to 0. \tag{3.3}$$

Let $\wp = I_P$ for any point $P \in \mathbb{X}$. Since

$$[\wp : I_{\mathbb{H}_u}] = \begin{cases} R, & \text{if } \ I_{\mathbb{H}_u} \in \wp \\ \wp, & \text{if } \ I_{\mathbb{H}_u} \notin \wp, \end{cases}$$

we have

$$\begin{aligned} [I_{\mathbb{X}} : I_{\mathbb{H}_u}] &= \bigcap_{P \in \mathbb{X}} [\wp : I_{\mathbb{H}_u}] = \left[\bigcap_{P \in \mathbb{Y}} [\wp : I_{\mathbb{H}_u}] \right] \cap \left[\bigcap_{P \in \mathbb{X} - \mathbb{Y}} [\wp : I_{\mathbb{H}_u}] \right] \\ &= \left[\bigcap_{P \in \mathbb{Y}} [\wp : I_{\mathbb{H}_u}] \right] \cap R = \bigcap_{P \in \mathbb{Y}} [\wp : I_{\mathbb{H}_u}] = \bigcap_{P \in \mathbb{Y}} \wp = I_{\mathbb{Y}}. \end{aligned}$$

Hence we can rewrite the exact sequence (3.3) as

$$0 \to I_{\mathbb{Y}}(-1) \xrightarrow{\times I_{\mathbb{H}_u}} I_{\mathbb{X}} \to \frac{I_{\mathbb{X}} + I_{\mathbb{H}_u}}{I_{\mathbb{H}_u}} \to 0. \tag{3.4}$$

From the long exact sequence of cohomology (using the functor $- \otimes_R k$) applied to the exact sequence (3.4) and using the fact that $I_{\mathbb{X}}$ and $I_{\mathbb{Y}}$ are height n perfect ideals, and so $\operatorname{Tor}_s^R(I_{\mathbb{X}}, k) = \operatorname{Tor}_s^R(I_{\mathbb{Y}}, k) = 0$ for every $s \geqslant n$, we obtain

$$\begin{array}{llllll} & & & 0 & \to & \operatorname{Tor}_n^R(J, k) \\ \to & \operatorname{Tor}_{n-1}^R(I_{\mathbb{Y}}, k)(-1) & \to & \operatorname{Tor}_{n-1}^R(I_{\mathbb{X}}, k) & \to & \operatorname{Tor}_{n-1}^R(J, k) \\ & & & \vdots & & \\ \to & \operatorname{Tor}_i^R(I_{\mathbb{Y}}, k)(-1) & \to & \operatorname{Tor}_i^R(I_{\mathbb{X}}, k) & \to & \operatorname{Tor}_i^R(J, k) \\ & & & \vdots & & \\ \to & \operatorname{Tor}_1^R(I_{\mathbb{Y}}, k)(-1) & \to & \operatorname{Tor}_1^R(I_{\mathbb{X}}, k) & \to & \operatorname{Tor}_1^R(J, k) \\ \to & I_{\mathbb{Y}} \otimes_R k(-1) & \xrightarrow{\times I_{\mathbb{H}_u} \otimes 1} & I_{\mathbb{X}} \otimes_R k & \to & J \otimes_R k \\ \to & 0. & & & & \end{array} \tag{3.5}$$

Notice that by (3.4), we obtain that

$$\mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{H}_u}),t)=\begin{cases}1 & \text{for } t=0,\\ \mathbf{H}(\mathbb{X},t)-\mathbf{H}(\mathbb{Y},t-1) & \text{for } t\geqslant 1.\end{cases}$$

Hence it follows from Remark 2.5 that

$$\mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{H}_u}),t)=\mathbf{H}(\mathbb{X}_u,t) \qquad \text{for all} \quad t\geqslant 0, \text{ i.e., } I_{\mathbb{X}}+I_{\mathbb{H}_u}=I_{\mathbb{X}_u}.$$

Thus

$$I_{\mathbb{X}_u}/I_{\mathbb{H}_u}=J.$$

Now consider the exact sequence

$$\begin{array}{c} 0\to I_{\mathbb{H}_u}\to I_{\mathbb{X}_u}\to I_{\mathbb{X}_u}/I_{\mathbb{H}_u}\to 0.\\ \qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad J \end{array} \tag{3.6}$$

From the long exact sequence of cohomology (using the functor $-\otimes_R k$ again), but now applied to the sequence (3.6)) we obtain

$$\begin{array}{ccccc}
 & & 0 & \to & \mathrm{Tor}_n^R(J,k)\\
\to \mathrm{Tor}_{n-1}^R(I_{\mathbb{H}_u},k) & \to & \mathrm{Tor}_{n-1}^R(I_{\mathbb{X}_u},k) & \to & \mathrm{Tor}_{n-1}^R(J,k)\\
 & & \vdots & & \\
\to \mathrm{Tor}_i^R(I_{\mathbb{H}_u},k) & \to & \mathrm{Tor}_i^R(I_{\mathbb{X}_u},k) & \to & \mathrm{Tor}_i^R(J,k)\\
 & & \vdots & & \\
\to \mathrm{Tor}_1^R(I_{\mathbb{H}_u},k) & \to & \mathrm{Tor}_1^R(I_{\mathbb{X}_u},k) & \to & \mathrm{Tor}_1^R(J,k)\\
\to I_{\mathbb{H}_u}\otimes_R k & \to & I_{\mathbb{X}_u}\otimes_R k & \to & J\otimes_R k \to 0.
\end{array}$$

Since $\mathrm{Tor}_i^R(I_{\mathbb{H}_u},k)=0$ for every $i\geqslant 1$, we obtain

$$\begin{aligned} &\mathrm{Tor}_n^R(J,k)=0\\ &\mathrm{Tor}_i^R(J,k)\simeq \mathrm{Tor}_i^R(I_{\mathbb{X}_u},k) \qquad \text{for every} \quad i=2,\ldots,n-1,\end{aligned} \tag{3.7}$$

and that

$$\begin{array}{ccccc}
0 & \to & \mathrm{Tor}_1^R(I_{\mathbb{X}_u},k) & \to & \mathrm{Tor}_1^R(J,k)\\
\to I_{\mathbb{H}_u}\otimes_R k & \to & I_{\mathbb{X}_u}\otimes_R k & \to & J\otimes_R k \to 0
\end{array}$$

is exact.

Since $\dim_k I_{\mathbb{H}_u} \otimes_R k + \dim_k J \otimes_R k = \dim_k I_{\mathbb{X}_u} \otimes_R k$, it follows that

$$0 \to I_{\mathbb{H}_u} \otimes_R k \to I_{\mathbb{X}_u} \otimes_R k \to J \otimes_R k \to 0$$

is exact. Thus $\operatorname{Tor}_1^R(J, k) \simeq \operatorname{Tor}_1^R(I_{\mathbb{X}_u}, k)$ and so (3.7) is also valid for $i = 1$.

Moreover, since $\dim_k I_{\mathbb{Y}} \otimes_R k(-1) + \dim_k J \otimes_R k = \dim_k I_{\mathbb{X}} \otimes_R k$, we also get that

$$0 \to I_{\mathbb{Y}} \otimes_R k(-1) \xrightarrow{\times I_{\mathbb{H}_u} \otimes 1} I_{\mathbb{X}} \otimes_R k \to J \otimes_R k \to 0$$

is exact.

Thus we can rewrite the long exact sequence (3.5) as

$$\begin{array}{ccccccc} 0 \to \operatorname{Tor}_{n-1}^R(I_{\mathbb{Y}}, k)(-1) & \to & \operatorname{Tor}_{n-1}^R(I_{\mathbb{X}}, k) & \to & \operatorname{Tor}_{n-1}^R(I_{\mathbb{X}_u}, k) \\ & & \vdots & & \\ \to \operatorname{Tor}_j^R(I_{\mathbb{Y}}, k)(-1) & \to & \operatorname{Tor}_j^R(I_{\mathbb{X}}, k) & \to & \operatorname{Tor}_j^R(I_{\mathbb{X}_u}, k) \\ & & \vdots & & \\ \to \operatorname{Tor}_1^R(I_{\mathbb{Y}}, k)(-1) & \to & \operatorname{Tor}_1^R(I_{\mathbb{X}}, k) & \to & \operatorname{Tor}_1^R(I_{\mathbb{X}_u}, k) \\ \to \quad 0. & & & & \end{array} \tag{3.8}$$

Let φ_j $(2 \leqslant j \leqslant n-1)$ be the connecting morphisms from $\operatorname{Tor}_j^R(I_{\mathbb{X}_u}, k)$ to $\operatorname{Tor}_{j-1}^R(I_{\mathbb{Y}}, k)(-1)$ in Eq. (3.8), i.e.,

$$\cdots \to \operatorname{Tor}_j^R(I_{\mathbb{X}_u}, k) \xrightarrow{\varphi_j} \operatorname{Tor}_{j-1}^R(I_{\mathbb{Y}}, k)(-1) \to \cdots.$$

CLAIM. $\varphi_j = 0$ *for every* $2 \leqslant j \leqslant n-1$.

Proof of Claim. Since $\sigma(\mathbb{Y}) = \sigma(\mathbb{X}_{u-1})$, we have that

$$[\operatorname{Tor}_i^R(I_{\mathbb{Y}}, k)]_t = 0 \qquad \text{for} \quad t > \sigma(\mathbb{X}_{u-1}) + i.$$

Now

$$[\varphi_j]_t \colon [\operatorname{Tor}_j^R(I_{\mathbb{X}_u}, k)]_t \to [\operatorname{Tor}_{j-1}^R(I_{\mathbb{Y}}, k)]_{t-1},$$

so, if $t-1 > \sigma(\mathbb{X}_{u-1}) + (j-1)$, i.e., if $t > \sigma(\mathbb{X}_{u-1}) + j$, we get that $\operatorname{Im}[\varphi_j]_t = 0$.

Now, suppose $t \leqslant \sigma(\mathbb{X}_{u-1}) + j$. Since $\sigma(\mathbb{X}_{u-1}) < \alpha(\mathbb{X}_u)$, we have $t < \alpha(\mathbb{X}_u) + j$, i.e., $t \leqslant \alpha(\mathbb{X}_u) + j - 1 = \alpha_{u0} + j - 1 = (a_{u0} - 1) + j$. But we know $I_{\mathbb{X}_u}$ has no generators in degree $\leqslant a_{u0} - 1$ and so $[\operatorname{Tor}_j^R(I_{\mathbb{X}_u}, k)]_t = 0$ for $t \leqslant (a_{u0} - 1) + j$. Thus $\operatorname{Im}[\varphi_j]_t = 0$ for every t.

From the claim, we see that

$$0 \to \operatorname{Tor}_j^R(I_{\mathbb{Y}}, k)(-1) \to \operatorname{Tor}_j^R(I_{\mathbb{X}}, k) \to \operatorname{Tor}_j^R(I_{\mathbb{X}_u}, k) \to 0 \tag{3.9}$$

is split exact for every $1 \leqslant j \leqslant n-1$. Therefore

$$\mathscr{F}_j = \mathscr{D}_j(-1) \oplus \mathscr{E}_j$$

for every $2 \leqslant j \leqslant n$. Since the resolutions of $I_{\mathbb{X}_u}$ and $\mathbb{Y}$ were minimal and these sequences split,

$$0 \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \to \mathscr{F}_j \to \cdots \to \mathscr{F}_1 \to R \to R/I_{\mathbb{X}} \to 0$$

is a minimal resolution. This completes the proof of the theorem. ∎

By induction (see [12] for the case $n=2$), we have as an immediate corollary.

COROLLARY 3.7. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$. The graded Betti numbers in a minimal free resolution of $R/I_{\mathbb{X}}$ depend only on Type($\mathbb{X}$).*

4. EXTREMAL RESOLUTIONS

Let I be a perfect homogeneous ideal of a polynomial ring $R = k[x_0, ..., x_n]$ over a field k and let

$$0 \to F_n \to \cdots \to F_0 \to R \to R/I \to 0$$

be a minimal free resolution of the ideal I.

Let T_R be the monoid of terms in $x_0, ..., x_n$; we consider, in T_R, the total ordering defined by

$$x_0^{a_0} \cdots x_n^{a_n} > x_0^{b_0} \cdots x_n^{b_n}$$

if the first nonzero coordinate of the vector

$$\left(\sum_{i=0}^{n} (a_i - b_i), a_0 - b_0, ..., a_n - b_n \right)$$

is positive. This ordering is commonly known as the *degree-lexicographic* order. A *lex-segment* is a ordered sequence of terms $s_1 > \cdots > s_r$ with the property that any term s for which $s_1 > s > s_r$ is in the given sequence where $\deg s_1 = \cdots = \deg s_r$ and s_1 is the largest term of that degree.

Definition 4.1. Let $I = \bigoplus_{t \geqslant 0} I_t$ be a graded ideal of R. We say that I is a *lex-segment ideal* if for every $t \geqslant 0$, I_t is generated (as a vector space) by a lex-segment.

By analogy with the discussion in [13], we would like to describe a very special k-configuration of points in $\mathbb{P}^n$.

Let $\mathcal{T} = (\mathcal{T}_1, ..., \mathcal{T}_u)$ be the *Type vector* of some k-configuration in $\mathbb{P}^n$. We want to describe a *standard k-configuration*, $\mathbb{X}$ in $\mathbb{P}^n$, such that $Type(\mathbb{X}) = \mathcal{T}$.

As mentioned, we did that for the case when $\mathcal{T}$ is the *Type*, of a k-configuration in $\mathbb{P}^3$. We will make the description by induction on n, starting with the explicit case $n = 3$ from [13].

Consider the affine space $\mathbb{A}^n \subset \mathbb{P}^n$ given by $x_0 \neq 0$ and (abusively) let the co-ordinates of this affine space be denoted $x_1, ..., x_n$.

Consider the planes $\pi_0 : x_1 = 0$, $\pi_1 : x_1 = 1, ..., \pi_{u-1} : x_1 = (u-1)$ and identify these planes with $\mathbb{A}^{n-1}$ where, if $x_1 = \alpha$, then

$$(\alpha, x_2, ..., x_n) \to (x_2, ..., x_n).$$

Then we place the standard k-configuration in $\mathbb{P}^{n-1}$ of *Type* $\mathcal{T}_1$ (by induction) in the plane π_{u-1}, the standard k-configuration in $\mathbb{P}^{n-1}$ of *Type* $\mathcal{T}_2$ in the plane π_{u-2}, and, continuing, the standard k-configuration in $\mathbb{P}^{n-1}$ of *Type* $\mathcal{T}_u$ in π_0. We call the resulting collection of points in $\mathbb{A}^n \subseteq \mathbb{P}^n$, a *standard k-configuration* of points in $\mathbb{P}^n$ of *Type* $\mathcal{T}$.

Example 4.2. A standard k-configuration in $\mathbb{P}^3$ of type ((1), (1, 3), (1, 2, 3, 5)) is shown in Fig. 5.

Let $S = R/(x_0) \simeq k[x_1, ..., x_n]$ $(n \geqslant 3)$.

Theorem 4.3. *Let $\mathbb{X}$ be a standard k-configuration in $\mathbb{P}^n$ of Type($\mathbb{X}$) and let $J = (I_{\mathbb{X}} + (x_0))/(x_0) \subset S$. Then J is a lex-segment ideal in S.*

Proof. The proof relies heavily on the work done in [8]. The reader is advised to consult that paper (especially Example 2.4) to understand what motivates this proof.

Suffice it to say that if we consider $\mathbb{A}^n(k)$ as $\mathbb{P}^n(k) - \mathcal{V}(x_0)$ and the lattice points of the positive "octant" in $\mathbb{A}^n(k)$ as corresponding to the monomials of $S = k[x_1, ..., x_n]$ (where $(a_1, ..., a_n) \leftrightarrow x_1^{a_1} \cdots x_n^{a_n}$) then if $I_{\mathbb{X}}$, $I_{\mathbb{X}} \subseteq R = k[x_0, ..., x_n]$, is the ideal of the points in a standard k-configuration in $\mathbb{P}^n$ then the monomials which correspond to the lattice points *not* in $\mathbb{X}$ are a set of generators for $J = (I_{\mathbb{X}} + (x_0))/(x_0) \subseteq S$.

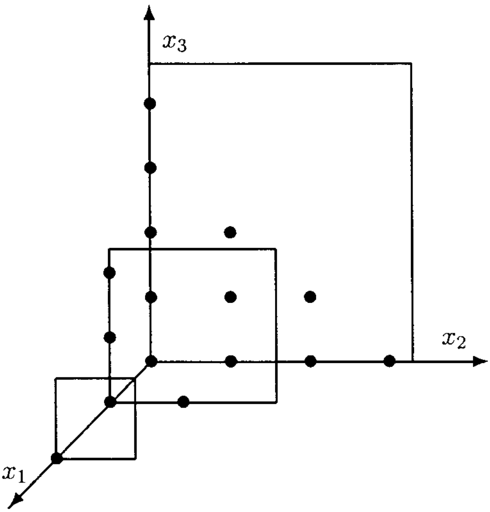

FIG. 5. Standard k-configuration in $\mathbb{P}^3$ of type ((1), (1, 3)(1, 2, 3, 5)).

So, to prove the theorem it suffices to prove that the ideal generated by the monomials which correspond to the points *not* in a standard k-configuration, generate a lex-segment ideal.

We shall prove the theorem by double induction on n and u. If $n=3$, then the theorem holds by Theorem 4.3 in [13].

Now assume $n>3$. If $u=1$, then $\mathbb{X}$ is a standard k-configuration in $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. Hence we are done in this case by induction on n. Assume $u>1$.

Let $\mathbb{X}_1, ..., \mathbb{X}_u$ be the sub-k-configurations of $\mathbb{X}$ in $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ where $u=\alpha(\mathbb{X})$. Since each $\mathbb{X}_i$ is a k-configuration in $\mathbb{P}^{n-1}$, there exist sub-k-configurations $\mathbb{X}_{(i,1)}, ..., \mathbb{X}_{(i,\ell_i)}$ of $\mathbb{X}_i$ in $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ where $\ell_i = \alpha(\mathbb{X}_i)$ for every $1 \leqslant i \leqslant u$.

Let

$$\circ = (u-i, \alpha(\mathbb{X}_i) - m_i, \alpha_2, ..., \alpha_{n-2}, \gamma)$$

represent a point in $\mathbb{A}^n$, which is not in $\mathbb{X}_i$, where $i>1$ and

$$\alpha_i \geqslant 0 \qquad \text{for every} \quad i=2, ..., n-2,$$

$$1 \leqslant m_i \leqslant \alpha(\mathbb{X}_i) \qquad \text{and} \qquad \gamma \geqslant \sigma(\mathbb{X}_{im_i}).$$

Consider the line which is the intersection of the planes

$$\begin{aligned}
x_1 &= u - i,\\
x_2 &= \alpha(\mathbb{X}_i) - m_i,\\
x_3 &= \alpha_2,\\
&\ \vdots\\
x_{n-2} &= \alpha_{n-2},\\
x_1 + x_2 + \cdots + x_n &= (u-i) + (\alpha(\mathbb{X}_i) - m_i) + \sum_{i=2}^{n-2} \alpha_i + \gamma.
\end{aligned} \tag{4.1}$$

Let $\mathscr{S}_{im_i}$ be the set of lattice points in $\mathbb{A}^n$ which satisfy the system (4.1) and whose components are all nonnegative.

Claim. *For every point $P \in \mathscr{S}_{im_i}$, $P \notin \mathbb{X}$.*

Proof of Claim. Assume $u = 1$. Then $\mathbb{X}$ is a standard k-configuration in $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ by induction on n. Hence we are done in this case. Assume $u > 1$ and for every $P \in \mathscr{S}_{im_i}$, $P \notin \mathbb{X}$ where $i > 1$. Then, in particular, the point

$$\bullet = \left(u - i, \alpha(\mathbb{X}_i) - 1, 0, \ldots, 0, \sum_{i=2}^{n-2} \alpha_i - m_i + 1 + \gamma, 0 \right)$$

is not in $\mathbb{X}_i$. To prove the claim, it suffices to show that the point

$$\diamondsuit = \left(u - (i-1), 0, 0, \ldots, 0, \alpha(\mathbb{X}_i) + \sum_{i=2}^{n-2} \alpha_i - m_i + \gamma - 1 \right)$$

does not belong to $\mathbb{X}_{i-1}$. However, since

$$\begin{aligned}
&\alpha(\mathbb{X}_i) + \sum_{i=2}^{n-2} \alpha_i - m_i + \gamma - 1\\
&\quad \geqslant (\alpha(\mathbb{X}_i) - 1) + (\gamma - m_i) \qquad \left(\because \sum_{i=2}^{n-2} \alpha_i \geqslant 0 \right)\\
&\quad \geqslant (\alpha(\mathbb{X}_i) - 1) + (\sigma(\mathbb{X}_{im_i}) - m_i)\\
&\quad \geqslant \alpha(\mathbb{X}_i) - 1 \qquad (\because \sigma(\mathbb{X}_{im_i}) - m_i \geqslant 0)\\
&\quad > \sigma(\mathbb{X}_{i-1}) - 1,
\end{aligned}$$

$\diamondsuit \notin \mathbb{X}_{i-1}$, i.e., $\diamondsuit \notin \mathbb{X}$. This completes the proof the claim, and so the proof of the theorem. ∎

Putting together the work of A. Bigatti [2], H. Hulett [17], K. Pardue [18], and Theorem 4.3, we obtain the following:

THEOREM 4.4. *Let k be an infinite field of arbitrary characteristic and let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ of Type*

$$\mathscr{T} = Type(\mathbb{X}).$$

Let $\mathbf{H}_{\mathscr{T}}$ be the Hilbert function of $\mathbb{X}$.

The resolution of $I_{\mathbb{X}}$ given in Theorem 3.6 is the extremal resolution possible for $\Delta\mathbf{H}_{\mathscr{T}}$.

5. COMPLEMENTARY CONFIGURATIONS IN $\mathbb{P}^N$

Recall that a finite complete intersection set of points $\mathbb{Z}$ in $\mathbb{P}^n$ is said to be a *basic configuration* in $\mathbb{P}^n$ (see [15]) if there exist integers $r_1, ..., r_n$ and distinct hyperplanes $\mathbb{L}_{ij}$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r_i)$ such that

$$\mathbb{Z} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_n \text{ as schemes,} \qquad \text{where} \quad \mathbb{H}_i = \mathbb{L}_{i1} \cup \cdots \cup \mathbb{L}_{ir_i}.$$

In this case $\mathbb{Z}$ is said to be of type $(r_1, ..., r_n)$.

DEFINITION 5.1. Let $\mathbb{Z}$ be a basic configuration in $\mathbb{P}^n$ and $\mathbb{X}$ a k-configuration in $\mathbb{P}^n$ which is contained in $\mathbb{Z}$. We call $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$ a *complementary configuration* in $\mathbb{P}^n$. (In our earlier papers [12, 13], we used the expression *weak-k-configuration* for this notion.)

EXAMPLE 5.2. Notice that (2, 3, 5) is the *Type* of a k-configuration in $\mathbb{P}^2$. If we embed a standard k-configuration of *Type* (2, 4, 5) in a basic configuration of type (3, 7), then the complementary configuration is a k-configuration of *Type* (2, 3, 5).

However, if we take a k-configuration in $\mathbb{P}^2$ of *Type* (3, 5) and embed it in a basic configuration of type (4, 5), then the complementary configuration has 2 points on one line, 5 points on a second line, and 5 points on a third line. So, a complementary configuration need not be a k-configuration.

Let $\mathbb{X}$ be k-configuration in $\mathbb{P}^n$ which is contained in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^n$ of type $(\alpha_1, ..., \alpha_n)$ where $\alpha_1 = \alpha(\mathbb{X}) = u$. Let $\mathbb{Z}_i$ be the points of $\mathbb{Z}$ on the hyperplane $\mathbb{H}_i$. Let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$.

THEOREM 5.3. *Let $\mathbb{Y}$ be as above and $I_{\mathbb{Y}}$ be the ideal of $\mathbb{Y}$. Then*

$$v(I_{\mathbb{Y}}) = v(\bar{I}_{\mathbb{Y}_1}) + v(\bar{I}_{\mathbb{Y}_2}) + \cdots + v(\bar{I}_{\mathbb{Y}_{\alpha_1}}) - (\alpha_1 - 1)\, n \tag{5.1}$$

and

$$\Delta(I_{\mathbb{Y}}) = \{\alpha_1, \bar{\Delta}(\bar{I}_{\mathbb{Y}_{\alpha_1}}) + \alpha_1 - 1, \bar{\Delta}(\bar{I}_{\mathbb{Y}_{\alpha_1 - 1}}) + \alpha_1 - 2, ..., \bar{\Delta}(\bar{I}_{\mathbb{Y}_2}) + 1, \Delta(\bar{I}_{\mathbb{Y}_1})\}, \tag{5.2}$$

where $\bar{I}_{\mathbb{Y}_j} = [I_{\mathbb{Y}_j} + I_{\mathbb{H}_j}]/I_{\mathbb{H}_j} \subset R/I_{\mathbb{H}_j}$ *for every* $1 \leqslant j \leqslant \alpha_1$ *where*

$$\bar{\Delta}(\bar{I}_{\mathbb{Y}_j}) = \Delta(\bar{I}_{\mathbb{Y}_j}) - \{\alpha_2, ..., \alpha_n\}$$

for every $2 \leqslant j \leqslant \alpha_1$.

Proof. Let $\mathbb{X}_i$ be as in Definition 2.4. Let $\mathbb{Y}_i := \mathbb{Z}_i - \mathbb{X}_i$ for every $i = 1, ..., \alpha_1$. Then $\mathbb{Y} = \bigcup_{i=1}^{\alpha_1} \mathbb{Y}_i$.

Let

$$\mathbb{Z}' = \bigcup_{i=1}^{\alpha_1 - 1} \mathbb{Z}_i, \qquad \mathbb{X}' = \bigcup_{i=1}^{\alpha_1 - 1} \mathbb{X}_i, \qquad \mathbb{Y}' = \bigcup_{i=1}^{\alpha_1 - 1} \mathbb{Y}_i,$$

$$\sigma = \sigma(\mathbb{Z}) = \sum_{i=1}^{n} \alpha_i - (n-1),$$

and

$$\sigma' = \sigma(\mathbb{Z}_{\alpha_1}) = 1 + \sum_{i=2}^{n} \alpha_i - (n-1) = \sum_{i=2}^{n} \alpha_i - n + 2.$$

Then, by Theorem 3 (b) in [6] we have, for every $t \geqslant 0$,

$$\Delta\mathbf{H}(\mathbb{Z}, t) = \Delta\mathbf{H}(\mathbb{X}, t) + \Delta\mathbf{H}(\mathbb{Y}, \sigma - 1 - t) \tag{5.3}$$

$$\Delta\mathbf{H}(\mathbb{Z}_{\alpha_1}, t) = \Delta\mathbf{H}(\mathbb{X}_{\alpha_1}, t) + \Delta\mathbf{H}(\mathbb{Y}_{\alpha_1}, \sigma' - 1 - t),$$

and

$$\Delta\mathbf{H}(\mathbb{Z}', t-1) = \Delta\mathbf{H}(\mathbb{X}', t-1) + \Delta\mathbf{H}(\mathbb{Y}', \sigma - 1 - t).$$

Since $\Delta\mathbf{H}(\mathbb{Z}, t) = \Delta\mathbf{H}(\mathbb{Z}_{\alpha_1}, t) + \Delta\mathbf{H}(\mathbb{Z}', t-1)$ and $\Delta\mathbf{H}(\mathbb{X}, t) = \Delta\mathbf{H}(\mathbb{X}_{\alpha_1}, t) + \Delta\mathbf{H}(\mathbb{X}', t-1)$,

$$\Delta\mathbf{H}(\mathbb{Y}, \sigma-1-t) = \Delta\mathbf{H}(\mathbb{Y}_{\alpha_1}, \sigma'-1-t) + \Delta\mathbf{H}(\mathbb{Y}', \sigma-1-t)$$

by Eqs. (5.3). Let $s = \sigma - 1 - t$. Then $\sigma - \sigma' = \alpha_1 - 1$, and hence

$$\Delta\mathbf{H}(\mathbb{Y}, s) = \Delta\mathbf{H}(\mathbb{Y}_{\alpha_1}, s-\alpha_1+1) + \Delta\mathbf{H}(\mathbb{Y}', s).$$

It follows that

$$\mathbf{H}(\mathbb{Y}, s) = \mathbf{H}(\mathbb{Y}_{\alpha_1}, s-\alpha_1+1) + \mathbf{H}(\mathbb{Y}', s)$$

for every $s \geqslant 0$. Furthermore, using a similar idea as above, we get

$$\mathbf{H}\left(\bigcup_{i=1}^{\alpha_1-j} \mathbb{Y}_i, s\right) = \mathbf{H}(\mathbb{Y}_{\alpha_1-j}, s-(\alpha_1-j)+1) + \mathbf{H}\left(\bigcup_{i=1}^{\alpha_1-j-1} \mathbb{Y}_i, s\right)$$

for every $1 \leqslant j < \alpha_1 - 1$. Thus

$$\mathbf{H}(\mathbb{Y}, s) = \mathbf{H}(\mathbb{Y}_1, s) + \mathbf{H}(\mathbb{Y}'', s-1) \tag{5.4}$$

for such an s where $\mathbb{Y}'' = \bigcup_{i=2}^{\alpha_1} \mathbb{Y}_i$.

We shall prove the theorem by double induction on n and α_1. If $n = 2$, then $\mathbb{Y}$ is a k-configuration in $\mathbb{P}^2$, and hence we are done in this case by Theorem 2.6 in [12].

Now assume $n > 2$. If $\alpha_1 = 1$, then $\mathbb{Y}$ is a complementary configuration in $\mathbb{P}^{n-1}$ and hence we are done in this case by induction on n. Now suppose $\alpha_1 > 1$. Let $\mathbb{H}$ be the hyperplane in $\mathbb{P}^n$ which contains $\mathbb{Z}_1$ and $I_{\mathbb{H}} = (H)$ be the ideal of $\mathbb{H}$ in R.

Let $S = R/I_{\mathbb{H}}$ and $J = [I_{\mathbb{Y}} + I_{\mathbb{H}}]/I_{\mathbb{H}}$. Then

$$\frac{I_{\mathbb{Y}}}{I_{\mathbb{H}} \cdot [I_{\mathbb{Y}} : I_{\mathbb{H}}]} = \frac{I_{\mathbb{Y}}}{I_{\mathbb{H}} \cap I_{\mathbb{Y}}} \simeq \frac{I_{\mathbb{Y}} + I_{\mathbb{H}}}{I_{\mathbb{H}}} = J \subset S.$$

Hence we have an exact sequence of graded modules

$$\begin{array}{c} 0 \to [I_{\mathbb{Y}} : I_{\mathbb{H}}](-1) \xrightarrow{\times H} I_{\mathbb{Y}} \to \dfrac{I_{\mathbb{Y}} + I_{\mathbb{H}}}{I_{\mathbb{H}}} \to 0. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad J \end{array} \tag{5.5}$$

Since $[I_{\mathbb{Y}} : I_{\mathbb{H}}] = I_{\mathbb{Y}''}$, we can rewrite the exact sequence (5.5) as

$$0 \to I_{\mathbb{Y}''}(-1) \xrightarrow{\times H} I_{\mathbb{Y}} \to J \to 0. \tag{5.6}$$

It follows from Eqs. (5.4) and (5.6) that

$$\begin{aligned}\mathbf{H}(S/J, t) &= \begin{cases} 1, & \text{for} \quad t=0, \\ \mathbf{H}(R/I_{\mathbb{Y}}, t) - \mathbf{H}(R/I_{\mathbb{Y}''}, t-1), & \text{for} \quad t \geqslant 1, \end{cases} \\ &= \mathbf{H}(R/I_{\mathbb{Y}_1}, t),\end{aligned}$$

which implies that J is a saturated ideal, i.e., $I_{\mathbb{Y}} + I_{\mathbb{H}} = I_{\mathbb{Y}_1}$.

By induction on α, there exist F's $\in I_{\mathbb{Y}}$ such that $\bar{F}$'s $\in S$ are the minimal generators of J. Let K's be the minimal generators of $I_{\mathbb{Y}''}$ and form the set of L's by multiplying all of the K's by H.

CLAIM. $I_{\mathbb{Y}} = \langle F\text{'s}, L\text{'s} \rangle$.

Proof of Claim. Clearly, $\langle F\text{'s}, L\text{'s}\rangle \subseteq I_{\mathbb{Y}}$. Conversely, for every $M \in I_{\mathbb{Y}}$, $\bar{M} \in J$. Hence

$$M = \left(\sum a\text{'s } F\text{'s} \right) + HN$$

for some a's $\in R$ and $N \in R$. Since $N \in [I_{\mathbb{Y}} : I_{\mathbb{H}}] = I_{\mathbb{Y}''}$,

$$N = \sum b\text{'s } K\text{'s}.$$

Hence

$$\begin{aligned} M &= \left(\sum a\text{'s } F\text{'s} \right) + HN \\ &= \left(\sum a\text{'s } F\text{'s} \right) + H \left(\sum b\text{'s } K\text{'s} \right) \\ &= \left(\sum a\text{'s } F\text{'s} \right) + \left(\sum b\text{'s } HK\text{'s} \right) \\ &= \left(\sum a\text{'s } F\text{'s} \right) + \left(\sum b\text{'s } L\text{'s} \right) \\ &\in \langle F\text{'s}, L\text{'s} \rangle. \end{aligned}$$

Thus

$$I_{\mathbb{Y}} = \langle F\text{'s}, L\text{'s} \rangle. \tag{5.7}$$

Let $F_1, ..., F_n$ be the generators of the ideal $I_{\mathbb{Z}}$ of degrees $\alpha_1, ..., \alpha_n$ and let $F_1 = \prod_{i=1}^{\alpha_1} H_i$ where $(H_i) = I_{\mathbb{H}_i}$ for every $i = 1, ..., \alpha_1$. Then $F_2, ..., F_n \in I_{\mathbb{Y}_i}$ for such an i, which means that

$$F_2, ..., F_n \in I_{\mathbb{Y}_1} \cap \cdots \cap I_{\mathbb{Y}_{\alpha_1}}. \tag{5.8}$$

From Eqs. (5.7) and (5.8) and the construction of the generators of the ideal $I_{\mathbb{Y}}$, we obtain Eqs. (5.1) and (5.2), which completes the proof. ∎

6. SOME UNIMODAL GORENSTEIN SEQUENCES OF CODIMENSION $n+1$

In the last sections of this paper we would like to apply the results of the previous sections to the study of (artinian) graded Gorenstein rings.

It is standard (see [12, 14, 15, 21]) that if $I_{\mathbb{X}}$ and $I_{\mathbb{Y}}$ are the ideals of disjoint, $\mathbb{X}$ and $\mathbb{Y}$, sets of non-degenerate points in $\mathbb{P}^n$ for which $\mathbb{X} \cup \mathbb{Y} = \mathbb{Z}$ is a complete intersection, then $I_{\mathbb{X}} + I_{\mathbb{Y}}$ is an ideal in $R = k[x_0, ..., x_n]$ of height $n+1$ for which $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is (artinian) Gorenstein. In our case, $\mathbb{X}$ will be a standard k-configuration embedded in a basic configuration $\mathbb{Z}$ and $\mathbb{Y} = \mathbb{Z} - \mathbb{X}$.

We will use the following description of the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$.

THEOREM 6.1 (Theorem 2.1 in [15]). *Let $\mathbb{X}$ and $\mathbb{Y}$ be d-dimensional arithmetically Cohen–Macaulay closed subschemes of $\mathbb{P}^n$ which are geometrically linked, and put $s = \sigma(\mathbb{X} \cup \mathbb{Y}) - 2$. Then*

(1) *$R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is a d-dimensional Gorenstein standard G-algebra.*

(2) $\Delta^d \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), i) = \Delta^d \mathbf{H}(\mathbb{X}, i) + \Delta^d \mathbf{H}(\mathbb{X}, s-i) - e(\mathbb{X})$ *for all* $i = 0, 1, ...$

(3) $\sigma(R/(I_{\mathbb{X}} + I_{\mathbb{Y}})) = s+1.$

(4) *Assume that* $2\sigma(\mathbb{X}) \leqslant \sigma(\mathbb{X} \cup \mathbb{Y})$. *Then* $\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), i)$, $i = 0, 1, ...,$ *is a Gorenstein SI-sequence. Furthermore, we have*

$$\Delta^d \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), i) = \begin{cases} \Delta^d \mathbf{H}(\mathbb{X}, i), & \text{for all} \quad i = 0, 1, ..., s - \sigma(\mathbb{X}) + 1, \\ \Delta^d \mathbf{H}(\mathbb{X}, s-i), & \text{for all} \quad i = s - \sigma(\mathbb{X}) + 1, ..., s. \end{cases}$$

Using Macauley's Theorem, it is easy to show the following lemma, so we shall omit the proof here.

LEMMA 6.2. *Let $\mathbb{X}$ be a set of points in $\mathbb{P}^2$. If $\alpha = \alpha(\mathbb{X})$, then*

$$\Delta \mathbf{H}(\mathbb{X}, t) \geqslant \Delta \mathbf{H}(\mathbb{X}, t+1)$$

for $t \geqslant \alpha$.

LEMMA 6.3. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ $(n \geqslant 3)$ and let $\mathbb{X}_1, ..., \mathbb{X}_u$ be the sub-k-configurations of $\mathbb{X}$ in $\mathbb{P}^{n-1}$. Then*

$$\Delta\mathbf{H}(\mathbb{X}, t) \geqslant \Delta\mathbf{H}(\mathbb{X}, t+1)$$

for every $t \geqslant \alpha(\mathbb{X}_u)$.

Proof. We shall prove this lemma by double induction on n and u. Let $n=3$ and $u=1$. Then $\mathbb{X}$ is a k-configuration in $\mathbb{P}^2$ and hence we are done in this case by Lemma 6.2.

Assume $u>1$ and let $\mathbb{Y} = \bigcup_{i=1}^{u-1} \mathbb{X}_i$. Then $\mathbb{Y}$ is a k-configuration in $\mathbb{P}^3$ and $\alpha(\mathbb{Y}) = \alpha(\mathbb{X}) - 1$. Now $\alpha(\mathbb{X}_{u-1}) \leqslant \sigma(\mathbb{X}_{u-1}) < \alpha(\mathbb{X}_u)$. If $t \geqslant \alpha(\mathbb{X}_u)$, then $(t-1) \geqslant \alpha(\mathbb{X}_{u-1})$. Hence, by induction on u, we have

$$\Delta\mathbf{H}(\mathbb{Y}, t-1) \geqslant \Delta\mathbf{H}(\mathbb{Y}, t)$$

for every $t \geqslant \alpha(\mathbb{X}_u)$. Therefore, for every $t \geqslant \alpha(\mathbb{X}_u)$,

$$\begin{aligned}\Delta\mathbf{H}(\mathbb{X}, t) &= \Delta\mathbf{H}(\mathbb{X}_u, t) + \Delta\mathbf{H}(\mathbb{Y}, t-1)\\ &\geqslant \Delta\mathbf{H}(\mathbb{X}_u, t+1) + \Delta\mathbf{H}(\mathbb{Y}, t)\\ &= \Delta\mathbf{H}(\mathbb{X}, t+1),\end{aligned}$$

as we wanted to show.

Now suppose $n>3$ and $u=1$. Then $\mathbb{X}$ is a k-configuration in $\mathbb{P}^m$ where $m<n$. Hence this case holds by induction on n.

If $n>3$ and $u>1$, then we obtain the result by the same method as above. This completes the proof. ∎

LEMMA 6.4. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ $(n \geqslant 3)$, $\mathbb{Z}$ a basic configuration in $\mathbb{P}^n$ which contains $\mathbb{X}$, and $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$. Let $\mathbb{X}_i$ be as in Lemma* 6.3 *and $\alpha(\mathbb{X}) \geqslant 2$. Then*

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t) \leqslant \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t+1)$$

for every $t < \alpha(\mathbb{X}_u) - 1$.

Proof. $\mathbb{X}$ is contained in many basic configurations, so we will have to take some care in describing them all.

Notice however, by induction on n, that there is a "smallest" basic configuration containing $\mathbb{X}$. It's of type $(\beta_1, ..., \beta_n)$ where $\beta_1 = \alpha(\mathbb{X})$, $\beta_2 = \alpha(\mathbb{X}_u)$, ..., $\beta_n = \sigma(\mathbb{X})$ (the intermediate β_i's are slightly more complicated to describe in a compact way).

Now, let $\mathbb{Z}$ be any basic configuration of type $(\alpha_1, ..., \alpha_n)$ containing $\mathbb{X}$, and let $\mathbb{Y}$ be the complementary configuration. If $\mathbb{Z}'$ is any other basic configuration of type $(\alpha', ..., \alpha'_n)$ containing $\mathbb{X}$, $\mathbb{Y}'$ the complementary configuration, and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha'_i$, then, by Theorem 6.1 (2),

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t) = \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}'}), t)$$

for all t. So, without loss of generality, we can assume $\alpha_1 = \beta_1, ..., \alpha_{n-1} = \beta_{n-1}$, and $\alpha_n = \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n-1} \beta_i$. It is easy to see that $\alpha_n \geqslant \sigma(\mathbb{X})$ from the choices of α_i's.

Since $\sigma(\mathbb{X} \cup \mathbb{Y}) = \sum_{i=1}^{n} \alpha_i - (n-1)$, if we let $s = \sigma(\mathbb{X} \cup \mathbb{Y}) - 2$, then

$$s = \sum_{i=1}^{n} \alpha_i - (n-1) - 2 = \sum_{i=1}^{n} \alpha_i - n - 1.$$

Thus, for every $t < \alpha(\mathbb{X}_u)$,

$$\begin{aligned} s - t &= \sum_{i=1}^{n} \alpha_i - n - 1 - t \\ &= \alpha_1 + \alpha_2 + \sum_{i=3}^{n-1} \alpha_i + \alpha_n - n - 1 - t \\ &\geqslant 2 + \alpha(\mathbb{X}_u) + (n-3) + \sigma(\mathbb{X}) - n - 1 - \alpha(\mathbb{X}_u) + 1 \\ &= \sigma(\mathbb{X}) - 1. \end{aligned}$$

Hence $\mathbf{H}(\mathbb{X}, s-t) = e(\mathbb{X})$ for such a t, where $e(\mathbb{X})$ is the multiplicity of $\mathbb{X}$. So, by Theorem 6.1 (2),

$$\begin{aligned} \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t) &= \mathbf{H}(R/I_{\mathbb{X}}, t) + \mathbf{H}(R/I_{\mathbb{X}}, s-t) - e(\mathbb{X}) \\ &= \mathbf{H}(R/I_{\mathbb{X}}, t) + e(\mathbb{X}) - e(\mathbb{X}) \\ &= \mathbf{H}(R/I_{\mathbb{X}}, t). \end{aligned}$$

Thus

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t) = \mathbf{H}(R/I_{\mathbb{X}}, t) \leqslant \mathbf{H}(R/I_{\mathbb{X}}, t+1) = \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t+1)$$

for every $t < \alpha(\mathbb{X}_u) - 1$ and this completes the proof. ∎

THEOREM 6.5. *Let $\mathbb{X}$ be a k-configuration in $\mathbb{P}^n$ $(n \geqslant 3)$, $\mathbb{Z}$ a basic configuration in $\mathbb{P}^n$ which contains $\mathbb{X}$, and $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$. Then the ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ is a Gorenstein ideal of codimension $\leqslant n+1$ and the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is unimodal.*

Proof. By Remarque 1.4 in [19], $I_{\mathbb{X}} + I_{\mathbb{Y}}$ is a Gorenstein ideal of codimension $\leqslant n+1$.

We shall prove the rest of the theorem by double induction on n and $\alpha(\mathbb{X})$. Let $\mathbb{Z}$ be a basic configuration in $\mathbb{P}^n$ of type $(\alpha_1, ..., \alpha_n)$. Without loss of generality, assume $\alpha_1 = \alpha(\mathbb{X})$ and $\alpha_2 = \alpha(\mathbb{X}_u)$.

Suppose $n=3$ and $\alpha_1 = 1$. Then $I_{\mathbb{X}} + I_{\mathbb{Y}}$ is a Gorenstein ideal of codimension 3, and hence the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is unimodal by Theorem 4.1 in [22]. Now assume $\alpha_1 > 1$.

Since $\alpha(\mathbb{X}) \geqslant 2$,

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t) \leqslant \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t+1)$$

for every $t < \alpha(\mathbb{X}_u) - 1$ by Lemma 6.4. Hence it suffices to show that

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t) \leqslant \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t+1)$$

for every $\alpha(\mathbb{X}_u) - 1 \leqslant t < [\frac{s}{2}]$.

Let $\alpha(\mathbb{X}_u) - 1 \leqslant t < [\frac{s}{2}]$. Since $t < [\frac{s}{2}]$, $\alpha(\mathbb{X}_u) \leqslant t+1 \leqslant s-t$ and hence

$$\Delta\mathbf{H}(R/I_{\mathbb{X}}, t+1) \geqslant \Delta\mathbf{H}(R/I_{\mathbb{X}}, s-t) \tag{6.1}$$

by Lemma 6.3. It follows from Eq. (6.1) that

$$\mathbf{H}(R/I_{\mathbb{X}}, s-t) - \mathbf{H}(R/I_{\mathbb{X}}, s-t-1) \leqslant \mathbf{H}(R/I_{\mathbb{X}}, t+1) - \mathbf{H}(R/I_{\mathbb{X}}, t).$$

In other words,

$$\mathbf{H}(R/I_{\mathbb{X}}, t) + \mathbf{H}(R/I_{\mathbb{X}}, s-t) \leqslant \mathbf{H}(R/I_{\mathbb{X}}, t+1) + \mathbf{H}(R/I_{\mathbb{X}}, s-t-1) \tag{6.2}$$

for every $\alpha(\mathbb{X}) \leqslant t+1 \leqslant s-t$. This means, using (6.2) and Theorem 6.1 (2), that

$$\begin{aligned}\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t) &= \mathbf{H}(R/I_{\mathbb{X}}, t) + \mathbf{H}(R/I_{\mathbb{X}}, s-t) - e(\mathbb{X}) \\ &\leqslant \mathbf{H}(R/I_{\mathbb{X}}, t+1) + \mathbf{H}(R/I_{\mathbb{X}}, s-t-1) - e(\mathbb{X}) \\ &= \mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), t+1)\end{aligned}$$

for $\alpha(\mathbb{X}_u) - 1 \leqslant t < [\frac{s}{2}]$. Hence we are done in this case.

Now assume $n > 3$ and $\alpha(\mathbb{X}) = 1$. Then $I_{\mathbb{X}} + I_{\mathbb{Y}}$ is a Gorenstein ideal of codimension $\leqslant n$. Hence the theorem holds by induction on n. If $\alpha(\mathbb{X}) > 1$, then the result follows in the same way as above. This completes the proof. ▮

7. THE MINIMAL FREE RESOLUTIONS OF SOME GORENSTEIN IDEALS OF CODIMENSION $n+1$

As we saw in the last section, if we use the *liaison* construction of Gorenstein ideals, starting with a standard configuration, $\mathbb{X}$, inside a basic configuration, $\mathbb{Z}$, (and having complement $\mathbb{Y}$), then the Hilbert function of the Gorenstein ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is completely determined by the Hilbert function of $R/I_{\mathbb{X}}$ and the number $\sigma(\mathbb{Z})$, and is unimodal.

We would like to say that all the graded Betti numbers in the minimal free resolution of $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ are determined only by the graded Betti numbers of $R/I_{\mathbb{X}}$ and $\sigma(\mathbb{Z})$. We showed that this was true for $n=2$ in [12] and for $n=3$ in [13]. However, the arguments in [12, 13] depended upon us knowing enough about the resolution of $\mathbb{Y}$ which, unfortunately, we do not have for $n>3$.

We can, however, describe the minimal free resolution of $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ with some additional hypothesis on $\mathbb{X}$ and $\mathbb{Z}$. It would be interesting to know if we can avoid these extra hypotheses.

In fact, with the additional hypotheses we even get a stronger result, which is a slight generalization of a wonderful result, first discovered by M. Boij (see [3]). Since our proof is slightly different from that of Boij we include it here.

THEOREM 7.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be d-dimensional arithmetically Cohen–Macaulay closed subschemes of $\mathbb{P}^n$ which are geometrically linked. Assume that $2\sigma(\mathbb{X})\leqslant\sigma(\mathbb{X}\cup\mathbb{Y})=\sigma$. Then*

$$[\operatorname{Tor}_i^R(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}),k)]_j = [\operatorname{Tor}_i^R(R/I_{\mathbb{X}},k)]_j \oplus [\operatorname{Tor}^R_{n-d-i+1}(R/I_{\mathbb{X}},k)]_{\sigma-1+n-d-j}$$

for all i, j.

Proof. First we recall a standard result of linkage theory that $I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}$ is isomorphism to the canonical module $\omega_{\mathbb{X}}=\operatorname{Ext}_R^{n-d}(R/I_{\mathbb{X}},R)$ of $R/I_{\mathbb{X}}$. Hence there is a number ℓ such that

$$I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}=\operatorname{Ext}_R^{n-d}(R/I_{\mathbb{X}},R)(\ell).$$

CLAIM. $\ell=-(\sigma-1+n-d)$.

Proof of Claim. From the short exact sequence

$$\begin{array}{c} 0\to(I_{\mathbb{X}}+I_{\mathbb{Y}})/I_{\mathbb{X}}\to R/I_{\mathbb{X}}\to R/(I_{\mathbb{X}}+I_{\mathbb{Y}})\to 0, \\ \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}} \qquad\qquad\qquad\qquad\qquad\qquad\quad \end{array}$$

we get the usual long exact sequence

$$\begin{aligned} &0 \qquad \to \operatorname{Tor}^R_{n+1-d}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k) \\ &\to \operatorname{Tor}^R_{n-d}(I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}, k) \to \operatorname{Tor}^R_{n-d}(R/I_{\mathbb{X}}, k) \to \operatorname{Tor}^R_{n-d}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k) \\ &\to \qquad \cdots. \end{aligned}$$

Note that

$$\begin{aligned} \operatorname{Tor}^R_{n-d}(I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}, k) &= \operatorname{Tor}^R_{n-d}(\omega_{\mathbb{X}}(\ell), k) \\ &= k(\ell). \end{aligned}$$

Furthermore, using Theorem 6.1 (3), we can check that

$$\operatorname{Tor}^R_{n+1-d}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k) = k(-(\sigma-1+n-d)).$$

Hence we get $\ell = -(\sigma-1+n-d)$, as we claimed.

Thus we have

$$\begin{aligned} [\operatorname{Tor}^R_i(I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}, k)]_j &= [\operatorname{Tor}^R_i(\omega_{\mathbb{X}}, k)]_{j+\ell} \\ &= [\operatorname{Tor}^R_{n-d-i}(R/I_{\mathbb{X}}, k)]_{-\ell-j} \\ &= [\operatorname{Tor}^R_{n-d-i}(R/I_{\mathbb{X}}, k)]_{\sigma-1+n-d-j}. \end{aligned}$$

Let φ_i $(0 \leqslant i \leqslant n-d)$ be the map from $\operatorname{Tor}^R_i(I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}, k)$ to $\operatorname{Tor}^R_i(R/I_{\mathbb{X}}, k)$.

Claim. *$\varphi_i = 0$ for all i.*

Proof of Claim. It follows, from Theorem 6.1 (4), that

$$\text{the initial degree of } I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}} \geqslant \sigma(\mathbb{X}).$$

Hence

$$[\operatorname{Tor}^R_i(I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}, k)]_j = 0$$

for all $j < \sigma(\mathbb{X}) + i$. Furthermore we note that

$$[\operatorname{Tor}^R_i(R/I_{\mathbb{X}}, k)]_j = 0$$

for all $j \geqslant \sigma(\mathbb{X}) + i$. Thus we get $\varphi_i = 0$.

With the claim proved, we get that the long exact sequence of cohomology above breaks up into short exact sequences

$$0 \to \operatorname{Tor}^R_i(R/I_{\mathbb{X}}, k) \to \operatorname{Tor}^R_i(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k) \to \operatorname{Tor}^R_{i-1}(I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}, k) \to 0.$$

Thus,

$$\begin{aligned}
&[\operatorname{Tor}_i^R(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k)]_j \\
&\quad = [\operatorname{Tor}_i^R(R/I_{\mathbb{X}}, k)]_j \oplus [\operatorname{Tor}_{i-1}^R(I_{\mathbb{Y}}/I_{\mathbb{X}\cup\mathbb{Y}}, k)]_j \\
&\quad = [\operatorname{Tor}_i^R(R/I_{\mathbb{X}}, k)]_j \oplus [\operatorname{Tor}_{n-d-i+1}^R(R/I_{\mathbb{X}}, k)]_{\sigma-1+n-d-j}. \qquad \blacksquare
\end{aligned}$$

We then have the following consequence.

THEOREM 7.2. *Let $\mathbb{X}$ be a standard k-configuration in $\mathbb{P}^n$ where the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \to \mathscr{F}_j \to \cdots \to \mathscr{F}_1 \to R \to R/I_{\mathbb{X}} \to 0.$$

Let $\mathbb{Z}$ be a basic configuration of type $(\alpha_1, ..., \alpha_n)$ containing $\mathbb{X}$ and set $\mathbb{Y} = \mathbb{Z} - \mathbb{X}$.

Suppose that $2\sigma(\mathbb{X}) \leqslant \sigma(\mathbb{Z}) = \sigma$, then the minimal free resolution of $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is

$$0 \to \mathscr{H}_{n+1} \to \cdots \to \mathscr{H}_1 \to R \to R/(I_{\mathbb{X}}+I_{\mathbb{Y}}) \to 0,$$

where

$$\mathscr{H}_{n+1} = R(-(\alpha_1 + \cdots + \alpha_n)) = R(-(\sigma+(n-1))$$

and if $1 \leqslant i < n+1$ then

$$[\mathscr{H}_i]_j = [\mathscr{F}_i]_j \oplus [\mathscr{F}_{n+1-i}]_{\sigma+n-1-j}.$$

Notice that if we start with a given standard k-configuration, $\mathbb{X}$, there are only a few basic configurations $\mathbb{Z}$ which contain $\mathbb{X}$ for which the hypothesis of Theorem 7.2 does not hold. We explore that fact in the next example.

EXAMPLE 7.3. Let $\mathbb{X}$ be a standard k-configuration in $\mathbb{P}^3$ of *Type* $((1, 2), (3, 5, 6))$. Then the Hilbert function of $\mathbb{X}$ is

$$\mathbf{H}(\mathbb{X}, t): 1\ \ 4\ \ 9\ \ 12\ \ 15\ \ 17 \to .$$

Consider a basic configuration $\mathbb{Z}$ in $\mathbb{P}^3$ of type $(2, 3, 6)$ containing $\mathbb{X}$ and let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$. So $\mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), -)$ is

$$\mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), t): 1\ \ 4\ \ 9\ \ 10\ \ 10\ \ 9\ \ 4\ \ 1\ \ 0 \to .$$

by Theorem 6.1 (2).

Since $2\sigma(\mathbb{X})$ is not less than or equal to $\sigma(\mathbb{Z})$ we cannot use Theorem 7.2 to find the minimal free resolution of $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$.

In spite of that, we did find the resolution in [10], and it is

$$\begin{array}{c} 0 \\ \downarrow \\ R(-11) \\ \downarrow \\ R^4(-5)\oplus R(-6)\oplus R(-7)\oplus R^6(-8)\oplus R(-9) \\ \downarrow \\ R^{10}(-4)\oplus R^2(-5)\oplus R^2(-6)\oplus R^{10}(-7) \\ \downarrow \\ R(-2)\oplus R^6(-3)\oplus R(-4)\oplus R(-5)\oplus R^4(-6) \\ \downarrow \\ R \\ \downarrow \\ R/(I_{\mathbb{X}}+I_{\mathbb{Y}}) \\ \downarrow \\ 0 \end{array} \tag{$\dagger$}$$

Notice, however, that there is another way to get a Gorenstein ring with Hilbert function that of $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$.

Start with the Hilbert function

$$1\ 4\ 9\ 10 \rightarrow$$

It is the Hilbert function of a k-configuration $\mathbb{X}'$ in $\mathbb{P}^3$ of *Type* ((1, 2), (1, 2, 4)) and $\sigma(\mathbb{X}')=4$.

Choose a basic configuration $\mathbb{Z}'$ in $\mathbb{P}^3$ of type (2, 3, 6) which contains $\mathbb{X}'$ and let $\mathbb{Y}' := \mathbb{Z}' - \mathbb{X}'$.

Now $\sigma(\mathbb{Z}')=9$ and so $2\sigma(\mathbb{X}')\leqslant\sigma(\mathbb{Z}')$ and we can apply Theorem 7.2 to get the minimal free resolution of $R/(I_{\mathbb{X}'}+I_{\mathbb{Y}'})$. The resulting resolution has the same graded Betti numbers we saw above.

A careful investigation of the first example above shows that the ideal generated by the degree 4 piece of $I_{\mathbb{X}}+I_{\mathbb{Y}}$ is supported only at 8 points, so the first example is very different from the second, in spite of the fact that they both lead to the same Betti numbers.

We suspect that this is not an isolated example.

8. EXTREMAL GORENSTEIN RINGS AND THE WEAK LEFSCHETZ PROPERTY

We believe that the resolutions we have constructed for Gorenstein ideals in Section 7 are extremal for their Hilbert function, so we would like to explain our thinking on this and offer some evidence for our belief.

Let $\mathbf{H}$ be the Hilbert function of a Gorenstein artinian quotient of $R=k[x_0, ..., x_n]$. Let I be an ideal of R for which R/I is artinian Gorenstein and such that the Hilbert function of R/I is $\mathbf{H}$. Let $\mathscr{F}$ be a minimal free R-resolution of R/I, i.e.,

$$\mathscr{F}: 0 \to \mathscr{F}_n \to \cdots \to \mathscr{F}_1 \to R \to R/I_{\mathbb{X}} \to 0.$$

Write $\mathscr{F}_i = \bigoplus_{j=1}^{\beta_i(I)} R(-a_{ij})$, where $a_{i1} \leqslant a_{i2} \leqslant \cdots \leqslant a_{i\beta_i(I)}$.

Consider the multi-set $\mathfrak{B}_i(I) = \{a_{i1}, a_{i2}, ..., a_{i\beta_i(I)}\}$.

If J is another Gorenstein ideal for which $\mathbf{H}(R/J, -) = \mathbf{H}$ and R/J has minimal free resolution $\mathscr{G}$ we also have the multisets

$$\mathfrak{B}_i(J) = \{b_{i1}, b_{i2}, ..., b_{i\beta_i(J)}\}.$$

We put a partial ordering on the set of resolutions of artinian Gorenstein ideals with the same Hilbert function $\mathbf{H}$ by saying that $\mathscr{F} \leqslant \mathscr{G}$ if $\mathfrak{B}_i(I) \subseteq \mathfrak{B}_i(J)$ for all $i = 1, ..., n$.

The question we would like to pose is whether or not this set of resolutions has a unique maximal and/or minimal element.

When $n=1$ the answer is easy. There is only one resolution for a given $\mathbf{H}$!

When $n=2$ the answer is not easy but both a maximum and a minimum exist for every possible $\mathbf{H}$ (see [4]).

When $n>2$ nothing is known.

Our interest in this problem comes from the fact that in [12] we showed how to construct (when $n=2$), for every relevant $\mathbf{H}$, an example of a Gorenstein ring with maximal Betti numbers starting with a standard k-configuration in $\mathbb{P}^2$ and using the method of liaison we've described above.

Conjecture 8.1. Suppose $n \geqslant 3$ and let $\mathbf{H}$ be a Gorenstein Hilbert function which arises using the construction of Section 7 (note that this is not *all* Gorenstein Hilbert functions, since by Theorem 6.5 we only get unimodal Hilbert functions while Bernstein–Iarrobino have shown in [1] that there are non-unimodal Gorenstein Hilbert functions when $n \geqslant 4$).

First notice that by Lemma 3.2 of [16], the artinian Gorenstein rings we constructed by Theorem 7.2 have the *weak Lefschetz property* (see [16] for the definition). By [24], it is known that "most" artinian Gorenstein rings have the weak Lefschetz property.

If we let $\mathscr{W}L$ denote the class of artinian Gorenstein rings with the Weak Lefschetz property them we can prove Conjecture 8.1 for this class. To be more precise:

THEOREM 8.2. *Let* $\mathbf{H}$ *be a Gorenstein Hilbert function arising from the construction of Theorem 7.2 and suppose it was made using* $\mathbb{X}$, *a standard k-configuration in* $\mathbb{P}^n$ *and* $\mathbb{Z}$ *a basic configuration of type* $(\alpha_1, \ldots, \alpha_n)$ *where*

$$\sigma(\mathbb{Z}) - 2\sigma(\mathbb{X}) \geqslant n.$$

Then the resolution of Theorem 7.2 has the largest Betti numbers among all the Gorenstein rings of $\mathscr{W}L$ *which have Hilbert function* $\mathbf{H}$.

Before beginning the proof, let's establish some notation. Let $R/I = A = \bigoplus A_i$ be an artinian Gorenstein ring in $\mathscr{W}L$ and let $g \in A_1$ be an element with the property that $g: A_i \to A_{i+1}$ (multiplication by g) is of maximal rank for every i. Such an element is guaranteed to exist because A is in $\mathscr{W}L$. As is well-known, A then has a unimodal Hilbert function and we set

$$a(A) = \min\{i \mid \mathbf{H}(A, i) \geqslant \mathbf{H}(A, i+1)\},$$

$$b(A) = \min\{i \mid \mathbf{H}(A, i) > \mathbf{H}(A, i+1)\},$$

and

$$\sigma(A) = \min\{i \mid \mathbf{H}(A, i) = 0\}.$$

Let's suppose that $\mathbf{H}(A, 1) = n+1$ and so $A = R/I$ where R is as above. Let $G \in R_1$ be chosen so that the class of G in A is g.

LEMMA 8.3. *With the notation above, we have*

$$[\mathrm{Tor}_i^R(A, k)]_j = [\mathrm{Tor}_i^{R/(G)}(A/(g), k)]_j$$

for all $j \leqslant b(A)$.

Proof. First we show

$$[\mathrm{Tor}_1^R(A, R/(G))]_j = 0$$

for all $j \leqslant b(A)$. Since $G \in R_1$ is not a zero-divisor,

$$\cdots \to 0 \to R(-1) \xrightarrow{\times G} R \to R/(G) \to 0$$

is a minimal free resolution of R/(G). Hence we get a complex

$$\cdots \to 0 \to R(-1) \otimes A \xrightarrow{\times G \otimes 1} R \otimes A \to R/(G) \otimes A \to 0.$$

Thus

$$\begin{aligned}\operatorname{Tor}_1^R(A, R/(G)) &= \operatorname{Tor}_1^R(R/(G), A)\\ &= \operatorname{Ker}(R(-1) \otimes A \xrightarrow{\times G \otimes 1} R \otimes A)\\ &= \{a \in A(-1) \mid ga = 0\}.\end{aligned}$$

Therefore it follows that

$$[\operatorname{Tor}_1^R(A, R/(G))]_j = \operatorname{Ker}(A_{j-1} \xrightarrow{\times g} A_j).$$

Since (A, g) has the weak Lefschetz property, we have

$$\operatorname{Ker}(A_{j-1} \xrightarrow{\times g} A_j) = 0$$

for all $j \leqslant b(A)$. Hence

$$[\operatorname{Tor}_1^R(A, R/(G))]_j = 0$$

for all $j \leqslant b(A)$. Let

$$\mathscr{F}_{\cdot}: 0 \to F_{n+1} \to \cdots \to F_i \xrightarrow{d_i} \cdots F_1 \to R \xrightarrow{d_0} A \to 0$$

be a minimal free resolution of A. Then we get a complex

$$\begin{aligned}\mathscr{F}_{\cdot} \otimes_R R/(G): 0 \to F_{n+1} \otimes R/(G) &\to \cdots \to\\ F_i \otimes R/(G) &\xrightarrow{d_i \otimes 1} \cdots \to R/(G) \to A/(g) \to 0.\end{aligned}$$

Since $[\operatorname{Tor}_i^R(A, R/(G))] = 0$ for all $i \geqslant 2$, we have

$$\operatorname{Im} d_{i+1} \otimes 1 = \operatorname{Ker} d_i \otimes 1$$

for all $i \geqslant 2$. Furthermore we note that

$$\operatorname{Im} d_1 \otimes 1 = \operatorname{Ker} d_0 \otimes 1.$$

That is, the complex $\mathscr{F}_{\cdot} \otimes_R R/(G)$ is exact except for the case $i = 1$. But, since

$$[\operatorname{Tor}_1^R(A, R/(G))]_j = 0$$

for all $j \leqslant b(A)$, we have

$$[\operatorname{Im} d_2 \otimes 1]_j = [\operatorname{Ker} d_1 \otimes 1]_j$$

for all $j \leqslant b(A)$. Hence it turns out that, for every $j \leqslant b(A)$,

$$\begin{aligned}[\operatorname{Tor}_i^{R/(G)}(A/(g), k)]_j &= [H_i((\mathscr{F} \otimes_R R/(G)) \otimes_{R/(G)} k)]_j \\ &= [H_i(\mathscr{F} \otimes_R (R/(G) \otimes_{R/(G)} k)]_j \\ &= [\operatorname{Tor}_i^R(A, k)]_j. \quad \blacksquare\end{aligned}$$

COROLLARY 8.4. *With the notation above, assume that* $b(A) - a(A) \geqslant n$. *Then*

$$[\operatorname{Tor}_i^R(A, k)]_j$$
$$= \begin{cases} [\operatorname{Tor}_i^{R/(G)}(A(g), k)]_j, & \text{for all} \quad j \leqslant [(\sigma(A)+n)/2], \\ [\operatorname{Tor}_{n+1-i}^{R/(G)}(A(g), k)]_{\sigma(A)+n-j}, & \text{for all} \quad j > [(\sigma(A)+n)/2]. \end{cases}$$

Proof. Since $b(A) - a(A) \geqslant n$, we can check from $\sigma(A) = a(A) + b(A) + 1$ that

$$[(\sigma(A)+n)/2] \leqslant b(A).$$

Hence from Lemma 8.3, we have

$$[\operatorname{Tor}_i^R(A, k)]_j = [\operatorname{Tor}_i^{R/(G)}(A(g), k)]_j$$

for all $j \leqslant [(\sigma(A)+n)/2]$. By the symmetry among the Betti numbers of Gorenstein rings, we obtain

$$[\operatorname{Tor}_i^R(A, k)]_j = [\operatorname{Tor}_{n+1-i}^R(A, k)]_{\sigma(A)+n-j}.$$

If $j > [(\sigma(A)+n)/2]$, then $\sigma(A)+n-j \leqslant [(\sigma(A)+n)/2] \leqslant b(A)$. Hence from Lemma 8.3, we get

$$[\operatorname{Tor}_i^R(A, k)]_j = [\operatorname{Tor}_{n+1-i}^{R/(G)}(A/(g), k)]_{\sigma(A)+n-j},$$

for all $j > [(\sigma(A)+n)/2]$. $\blacksquare$

Proof of Theorem 8.2. Let $R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \in \mathscr{W}\mathscr{L}$ be an artinian Gorenstein ring constructed by Theorem 7.2, and $(A, g) \in \mathscr{W}\mathscr{L}$ an artinian Gorenstein ring with the Hilbert function $\mathbf{H}$ of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$. We will show that

$$\dim_k [\operatorname{Tor}_i^R(A, k)]_j \leqslant \dim_k [\operatorname{Tor}_i^R(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), k)]_j$$

for all i, j. From the symmetry of the Betti numbers of Gorenstein rings, we have

$$\begin{aligned}\dim_k [\mathrm{Tor}_i^R(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k)]_j \\ = \dim_k [\mathrm{Tor}_{n+1-i}^R(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k)]_{\sigma(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}))+n-j}\end{aligned}$$

and

$$\dim_k [\mathrm{Tor}_i^R(A, k)]_j = \dim_k [\mathrm{Tor}_{n+1-i}^R(A, k)]_{\sigma(A)+n-j}.$$

Furthermore we note that $\sigma(R/(I_{\mathbb{X}}+I_{\mathbb{Y}})) = \sigma(A)$. Hence it is enough to show that

$$\dim_k [\mathrm{Tor}_i^R(A, k)]_j \leqslant \dim_k [\mathrm{Tor}_i^R(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k)]_j$$

for all $j \leqslant [(\sigma(A)+n)/2]$. It follows, from Lemma 3.1 in [16], that

$$\begin{aligned}a(R/(I_{\mathbb{X}}+I_{\mathbb{Y}})) &= a(A) = \sigma(\mathbb{X})-1,\\ b(R/(I_{\mathbb{X}}+I_{\mathbb{Y}})) &= b(A) = \sigma(\mathbb{X}\cup\mathbb{Y})-\sigma(\mathbb{X})-1\end{aligned}$$

and

$$\sigma(R/(I_{\mathbb{X}}+I_{\mathbb{Y}})) = \sigma(A) = \sigma(\mathbb{X}\cup\mathbb{Y})-1.$$

Hence $b(A)-b(a) = \sigma(\mathbb{X}\cup\mathbb{Y})-2\sigma(\mathbb{X}) \geqslant n$, and we get $[(\sigma(A)+n)/2] \leqslant b(A)$. Thus by virtue of Theorem 7.2, we can check that

$$[\mathrm{Tor}_i^R(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}), k)]_j = [\mathrm{Tor}_i^R(R/I_{\mathbb{X}}, k)]_j$$

for all $j \leqslant [(\sigma(A)+n)/2]$. It follows from Lemma 8.3 that

$$[\mathrm{Tor}_i^R(A, k)]_j = [\mathrm{Tor}_i^{R/(G)}(A/(g), k)]_j$$

for all $j \leqslant [(\sigma(A)+n)/2]$. Also, since (A, g) has the weak Lefschetz property and $\mathbf{H}(A, i) = \mathbf{H}(\mathbb{X}, i)$ for all i, we get

$$\mathbf{H}(A/(g), i) = \Delta\mathbf{H}(\mathbb{X}, i)$$

for all $i \geqslant 0$. Hence, by virtue of Theorem 4.4 and Lemma 8.3, we have that

$$\dim_k [\mathrm{Tor}_i^{R/(G)}(A/(g), k)]_j \leqslant \dim_k [\mathrm{Tor}_i^R(R/I_{\mathbb{X}}, k)]_j$$

for all i, j. Thus it turns out that for each $i = 1, 2, ..., n$

$$\dim_k [\operatorname{Tor}_i^R(A, k)]_j \leqslant \dim_k [\operatorname{Tor}_i^R(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), k)]_j$$

for all $j \leqslant [(\sigma(A) + n)/2]$. ∎

From this theorem we can observe that the resolution of an artinian Gorenstein ring of $\mathscr{W}\mathscr{L}$ with Hilbert function $\mathbf{H}$ can be obtained from the resolution of Theorem 7.2 by canceling some terms.

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